1. Lecture 8

• Last time we studied the Berry connection associated with a spin-1/2 particle in a magnetic field

$$H = \vec{B} \cdot \vec{\sigma} \ . \tag{1.1}$$

The corresponding curvature is

$$\vec{V} = \vec{\nabla} \times \vec{A} = \frac{\vec{B}}{2B^3} \ , \tag{1.2}$$

where this is computed in the $|\uparrow\rangle$ state.

• We saw that we could embed the above system in $\mathcal{N} = (2,2)$ SUSY via the following Lagrangian

$$\mathcal{L}_{\rm mass} = -\int d^4\theta \bar{\Phi} e^{-2\bar{\theta}\sigma^a\theta m_a}\Phi = |\phi'|^2 + i\psi\bar{\psi}' - m^a m_a|\phi|^2 + m^a\psi\sigma_a\bar{\psi} \ , \tag{1.3}$$

where m^a plays the role of the x^a $SU(2)_R$ triplet (note that we have solved the EOM of F and set $F = 0$)... In particular, the mass parameters enter as a background gauge multiplet (in analogy with the background magnetic field in the above example).

• Performing the usual transformation back to the Hamiltonian formulation gave us

$$H = |\pi|^2 + m^a m_a|\phi|^2 + \bar{\psi} m^a\sigma_a\psi \ , \tag{1.4}$$

The Fermionic operators can be arranged in creation and annihilation operators and yield the space

$$|0\rangle \ , \quad \bar{\psi}_+|0\rangle \ , \quad \bar{\psi}_-|0\rangle \ , \quad \bar{\psi}_+\bar{\psi}_-|0\rangle \ , \tag{1.5}$$

where

$$\psi_\pm|0\rangle = 0 \ . \tag{1.6}$$

On the space $\{|0\rangle, \bar{\psi}_+\bar{\psi}_-|0\rangle\}$, $H_\psi = \bar{\psi} m^a\sigma_a\psi$ vanishes. However, on the $\{\bar{\psi}_-|0\rangle, \bar{\psi}_+|0\rangle\}$ subspace we have

$$H_\psi\bar{\psi}_\alpha|0\rangle = m^a\sigma_{a\alpha}^{\ \ \beta}\bar{\psi}_\beta|0\rangle \ . \tag{1.7}$$

• On the homework, you studied the most general massive Lagrangian

$$\begin{aligned} \mathcal{L} &= -\int d^4\theta\varphi^\dagger e^{-2\bar{\theta}\sigma_a\theta m^a}\varphi + \int d^2\theta\mu\varphi^2 - \int d^2\bar{\theta}\bar{\mu}\varphi^2 \\ &= |\phi'|^2 + i\psi\psi'^\dagger - m^2|\phi|^2 + m^a\psi\sigma_a\psi^\dagger - 4|\mu|^2|\phi|^2 + \mu\psi^2 - \bar{\mu}\psi^{\dagger 2} \ , \end{aligned} \tag{1.8}$$

where $m = \sqrt{m^a m_a}$. You showed that if $m \neq 0$ with $\mu = 0$, there is a SUSY vacuum. You also (optionally!) showed that if $\mu \neq 0$ with $m = 0$, there is a SUSY vacuum (while this does not hold for $m, \mu \neq 0$).

• Today we want to move on to $2+1$D from $0+1$D. We started with $\mathcal{N} = 2$ in $0+1$ and then moved on to $\mathcal{N} = (2,2)$. We used this latter algebra to learn about Berry's phase in SQM. But this latter algebra is also useful because it connects more smoothly with 3D.

• For the rest of the module we will mostly be concerned with the 3D $\mathcal{N} = 2$ algebra

$$\{Q_\alpha, \bar{Q}_\beta\} = 2\gamma^\mu_{\alpha\beta} P_\mu \ , \quad \gamma^\mu_{\alpha\beta} = \gamma^\mu_{\beta\alpha} \ , \quad \mu = 1, 2, 3 \ , \tag{1.9}$$

Comment: This algebra looks quite similar to the $\mathcal{N} = (2,2)$ SQM algebra, and we will see why, but note there are also a few differences: here α, β are spacetime spinor indices (as opposed to internal R-symmetry indices; note that in both cases, the symmetries in question do not commute with the supercharges)... Also, there is no longer just the Hamiltonian sitting on the RHS of (1.9). Instead, special relativity in 3D forces us to include momentum generators in the spatial directions as well. Also, note that the gamma matrices are symmetric, i.e., we have spin 1 or vector generators.

• There is now a $U(1)_R$ automorphism ($SU(2)_R$ is no longer present, it is replaced by a spacetime symmetry)... We will come back again and again to the important role played by $U(1)_R$ in the coming lectures.

• An aside on spinors: spinors are in the double cover of the space-time symmetry group. If we are in Euclidean space, then this is the double cover of $SO(3)$, i.e., $SU(2) = \mathrm{Spin}(3)$. If we are in Lorentzian space, then this is the double cover of $SO(2,1)$, i.e., $SL(2,\mathbb{R}) = \mathrm{Spin}(2,1)$. We will spend much of the remainder of this module in Lorentzian signature and take

$$\gamma^\mu_{\alpha\beta} = \left(\sigma^0, \sigma^1, \sigma^3\right) \ . \tag{1.10}$$

where

$$\sigma^0_{\alpha\beta} \;=\; -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \quad \sigma^1_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma^3_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{1.11}$$

The following generators generate $SL(2,R)$

$$\epsilon^{\beta\gamma}\sigma^0_{\alpha\gamma} = \sigma^{0\beta}_\alpha \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \quad \epsilon^{\beta\gamma}\sigma^1_{\alpha\gamma} = \sigma^{1\beta}_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \quad \epsilon^{\beta\gamma}\sigma^3_{\alpha\gamma} = \sigma^{3\beta}_\alpha = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \ . \tag{1.12}$$

Note that

$$(\gamma^\mu)_\alpha^\beta(\gamma^\nu)_\beta^\lambda = \eta^{\mu\nu}\delta_\alpha^\lambda + \epsilon^{\mu\nu\rho}(\gamma_\rho)_\alpha^\lambda \ . \tag{1.13}$$

The main difference is that $SO(2,1)$ has real 2-component spinors while $SU(2)$ does not... The supercharges form a complex 2 component spinor anyway, so can use either space-time symmetry group. See Polchinski volume II for a discussion of spinors in various dimensions...

• Both this 3D $\mathcal{N}=2$ algebra and the $\mathcal{N}=(2,2)$ SQM algebra can be obtained via dimensional reduction of the 4D $\mathcal{N}=1$ SUSY algebra.

• What is dimensional reduction? It is a process to start from some quantum system in d space-time dimensions and reduce it to a quantum system in $d-r<d$ space-time dimensions.

• Suppose these r dimensions form some compact manifold, $\mathcal{M}_r$ (e.g., $\mathcal{M}_r = T^r = S^1 \times \cdots \times S^1$). Suppose $\mathcal{M}_r$ has some characteristic length-scale, L (this could be the period of the circles in the T^r). Then, quantum mechanics tells us that $p = n_i/L$ for $n_i \in \mathbb{Z}$ (and $i = 1, \cdots, r$). As we take $L \to 0$, $p \to \infty$ and so too the energy... Therefore, in this limit, the only finite energy configurations are those that are independent of the extra dimensions... These have $n_i = 0$... Specializing to these modes that have no dependence on $\mathcal{M}_r$, we get the dimensional reduction. This is equivalent to setting momentum to zero in the internal dimensions....

• Note that symmetries like rotations of these internal dimensions become internal symmetries of the dimensionally reduced theory (will see this below).

• Let us return to 4D→3D SUSY. There are unfortunately many conventions at play here... To get to 3d $\mathcal{N}=2$, we start in 4D from the Wess and Bagger conventions

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}}P_\mu \ , \tag{1.14}$$

$$\begin{aligned}
\sigma^0_{\alpha\dot{\beta}} &= -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \quad \sigma^1_{\alpha\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma^2_{\alpha\dot{\beta}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \\
\sigma^3_{\alpha\dot{\beta}} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .
\end{aligned} \tag{1.15}$$

we have here both "dotted" (e.g., "$\dot{\alpha}$") and "undotted" (e.g., "α") spinor indices here. This is because $\text{Spin}(3,1) = SL(2,\mathbb{C})$, so twice as many types of spinor reps. Similarly, for the Euclidean case, $\text{Spin}(4) = SU(2) \times SU(2)$, so there are twice as many types of spinor reps.

• Let us now reduce by, say, setting to zero momentum in the x^2 direction of $\mathbb{R}^4 \to \mathbb{R}^3 \times S^1$ (where x^2 parameterizes the S^1). This means, we set $P_2 = 0$ and obtain the algebra we had for 3D $\mathcal{N} = 2$ (we drop the dotted index because in 3D there is no distinction between dotted and undotted).

• To get to our $\mathcal{N} = (2,2)$ SQM, we would start from a slightly different convention for our $\sigma^\mu_{\alpha\dot{\alpha}}$, e.g.,

$$\sigma^0_{\alpha\dot{\alpha}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma^1_{\alpha\dot{\alpha}} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \ , \quad \sigma^2_{\alpha\dot{\alpha}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \quad \sigma^3_{\alpha\dot{\alpha}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{1.16}$$

We would then set $P^3 = P^2 = P^1 = 0$. The spin group of the transverse 3D become generators of the $SU(2)_R$ symmetry of the quantum mechanics...

• Back to QFT: roughly, we should treat free quantum fields as operator valued functions of space obeying equal time commutation relations (in the Heisenberg picture)

$$[\varphi_a(x_i,t), \varphi_b(y_i,t)] = [\pi^a(x_i,t), \pi^b(y_i,t)] = 0 \ , \quad [\pi^b(x_i,t), \phi_a(y_i,t)] = -i\delta^{(2)}(x_i - y_i)\delta^b_a \ . \tag{1.17}$$

• Since we are studying objects that depend on both space and time, we should modify our superspace differential operators. They become

$$\mathcal{Q}_\alpha = \partial_{\theta^\alpha} + i\gamma^{\mu\beta}_\alpha \bar{\theta}_\beta \partial_\mu \ , \quad \bar{\mathcal{Q}}_\alpha = -\partial_{\bar{\theta}^\alpha} - i\gamma^{\mu\beta}_\alpha \theta_\beta \partial_\mu \ . \tag{1.18}$$

where we raise and lower with $\epsilon^{\alpha\beta}$ and $\epsilon_{\alpha\beta}$ as follows

$$\epsilon^{\alpha\beta}\theta_\beta = \theta^\alpha \ , \quad \epsilon^{\alpha\beta}\bar{\theta}_\beta = \bar{\theta}^\alpha \ , \quad \epsilon_{\alpha\beta}\theta^\beta = \theta_\alpha \ , \quad \epsilon_{\alpha\beta}\bar{\theta}^\beta = \bar{\theta}_\alpha \ . \tag{1.19}$$

Therefore, we have

$$\overline{(\psi\chi)} = -\bar{\chi}\bar{\psi} \ . \tag{1.20}$$

The SUSY covariant derivatives are now

$$\mathcal{D}_\alpha = \partial_{\theta^\alpha} - i\gamma^{\mu\beta}_\alpha \bar{\theta}_\beta \partial_\mu \ , \quad \bar{\mathcal{D}}_\alpha = -\partial_{\bar{\theta}^\alpha} + i\gamma^{\mu\beta}_\alpha \theta_\beta \partial_\mu \ . \tag{1.21}$$

These quantities satisfy

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta\} = -2i\gamma^\mu_{\alpha\beta}\partial_\mu \ , \quad \{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_\beta\} = 2i\gamma^\mu_{\alpha\beta}\partial_\mu \ . \tag{1.22}$$

Other useful identities include **(Exercise!)**

$$\mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha = \bar{\mathcal{D}}^\alpha\mathcal{D}_\alpha \ , \quad \{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \cdots = 0 \ . \tag{1.23}$$

We also have the SUSY integration definitions

$$\int d^2\theta\theta^2 = 1 \ , \quad \int d^2\bar{\theta}\bar{\theta}^2 = -1 \ , \quad \int d^4\theta\theta^2\bar{\theta}^2 = -1 \ . \tag{1.24}$$

• What are some representations of the 3D $\mathcal{N}=2$ SUSY algebra? Well, we again have our friend the chiral multiplet, which is a function of θ_α and $y^\mu = x^\mu - i\theta\gamma^\mu\bar{\theta}$

$$\begin{aligned}\Phi &= \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y) = \phi(x) - i\theta\gamma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{1}{4}\theta^2\bar{\theta}^2\partial^2\phi(x) + \sqrt{2}\theta\psi(x) \\ &+ \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi(x)\gamma^\mu\bar{\theta} + \theta^2 F(x) \ . \end{aligned}\tag{1.25}$$

Similarly, we have an anti-chiral multiplet

$$\begin{aligned}\bar{\Phi} &= \bar{\phi}(y) - \sqrt{2}\bar{\theta}\bar{\psi}(y) - \bar{\theta}^2\bar{F}(y) = \bar{\phi}(x) + i\theta\gamma^\mu\bar{\theta}\partial_\mu\bar{\phi}(x) - \frac{1}{4}\theta^2\bar{\theta}^2\partial^2\bar{\phi}(x) - \sqrt{2}\bar{\theta}\bar{\psi}(x) \\ &- \frac{i}{\sqrt{2}}\bar{\theta}^2\theta\gamma^\mu\partial_\mu\bar{\psi}(x) + \bar{\theta}^2\bar{F}(x) \ . \end{aligned}\tag{1.26}$$

Note that these multiplets satisfy $0 = \mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha\Phi = \bar{\mathcal{D}}^\alpha\mathcal{D}_\alpha\Phi$, and similarly for $\bar{\Phi}$.

• We also have our friend the $U(1)$ vector multiplet

$$V = \theta\gamma^\mu\bar{\theta}A_\mu - i\bar{\theta}\theta\sigma - i\theta^2\cdot\bar{\theta}\bar{\lambda} + i\bar{\theta}^2\theta\lambda - \frac{1}{2}\theta^2\bar{\theta}^2 D \ , \tag{1.27}$$

This representation with the lower components vanishing is sometimes called the "Wess-Zumino" (or WZ) gauge... Notice that now the gauge field transforms as a **3** of $SO(2,1)$ (or $SO(3)$ in Euclidean space) while the scalar is a singlet (this is the A_2 direction of the 4D gauge field). Where again we have used

$$V \to V + \frac{i}{2}(\Lambda - \bar{\Lambda}) \ , \tag{1.28}$$

with Λ chiral and $\bar{\Lambda}$ anti-chiral (i.e., $D_\alpha\Lambda = \bar{D}_\alpha\bar{\Lambda} = 0$).

• To get a field-strength, we want a gauge invariant superfield. Clearly, we get this from setting

$$\Sigma = \frac{i}{2}D\bar{D}V = \frac{i}{2}\bar{D}DV \ , \tag{1.29}$$

since $\bar{\mathcal{D}}\mathcal{D}\Lambda = \bar{\mathcal{D}}\mathcal{D}\bar{\Lambda} = \bar{\mathcal{D}}\mathcal{D}\bar{\Lambda} = \bar{\mathcal{D}}\mathcal{D}\bar{\Lambda} = 0$. Note that since $\mathcal{D}\bar{\mathcal{D}} = \bar{\mathcal{D}}\mathcal{D}$, we have

$$\mathcal{D}^2\Sigma = \bar{\mathcal{D}}^2\Sigma = 0 \ . \tag{1.30}$$

These are the equations satisfied by a conserved current superfield... A SUSY generalization of a conserved current

$$\Sigma = \sigma - \theta\bar{\lambda} + \bar{\theta}\lambda + \frac{1}{2}\theta\gamma^{\mu}\bar{\theta}\epsilon_{\mu\nu\rho}F^{\nu\rho} - i\theta\bar{\theta}D + \cdots \ , \tag{1.31}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The current $j_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho}$ is sometimes called a "topological current." It is an abelian ($U(1)$ current) that is conserved

$$\partial^\mu j_\mu = \epsilon_{\mu\nu\rho}\partial^\mu\partial^\nu A^\rho = 0 \ . \tag{1.32}$$

How does this current act? It is instructive to use the fact that a free photon and a free scalar are dual in 3D (in 4D, we have that a photon and a photon are dual)

$$\partial_\mu\phi = \epsilon_{\mu\nu\rho}F^{\nu\rho} \ . \tag{1.33}$$

Indeed, we have that $\partial^2\phi = 0$ is equivalent to $\epsilon_{\mu\nu\rho}\partial^\mu F^{\nu\rho} = 0$ and $\epsilon_{\mu\nu\rho}\partial^\nu\partial^\rho\phi \sim \partial^\nu F_{\mu\nu} = 0$.

• For a $U(1)$ gauge group, ϕ is a periodic scalar (i.e., $\phi \sim \phi + 2\pi\ell^{-\frac{1}{2}}$). This is because there can be non-trivial flux through two-cycles in a 3D spacetime, and the scalar winds through the dual 1 cycle (e.g., think of T^3). Note: the current in (1.33) is for a shift symmetry $\phi \to \phi + \kappa$.

• Toward the end of the module, we will, in some sense, see how this duality extends to interacting SUSY theories.

• More generally, a current superfield looks as follows

$$\mathcal{J} = J + i\theta j + i\bar{\theta}\bar{j} - \theta\gamma^\mu\bar{\theta}j_\mu + i\theta\bar{\theta}K + \cdots \ , \tag{1.34}$$

It is real and satisfies $D^2\mathcal{J} = \bar{D}^2\mathcal{J} = 0$. The component version of this is $\partial^\mu j_\mu = 0$. We see this as follows

$$\begin{aligned}
\partial^{\alpha\beta}(D_\alpha\bar{D}_\beta + D_\beta\bar{D}_\alpha)\mathcal{J} \ &\sim \ \{D^\alpha, \bar{D}^\beta\}(D_\alpha\bar{D}_\beta + D_\beta\bar{D}_\alpha)\mathcal{J} = (D^\alpha\bar{D}^\beta D_\alpha\bar{D}_\beta + \bar{D}^\beta D^\alpha D_\alpha\bar{D}_\beta \\
&+ \ D^\alpha\bar{D}^\beta D_\beta\bar{D}_\alpha + \bar{D}^\beta D^\alpha D_\beta\bar{D}_\alpha)\mathcal{J} = (D^\alpha\{\bar{D}^\beta, D_\alpha\}\bar{D}_\beta + \bar{D}^\beta[D^\alpha D_\alpha, \bar{D}_\beta] \\
&+ \ D^\alpha\{\bar{D}^\beta, D_\beta\}\bar{D}_\alpha + [\bar{D}^\beta, D^\alpha D_\beta]\bar{D}_\alpha)\mathcal{J} \\
&= \ (D^\alpha\{\bar{D}^\beta, D_\alpha\}\bar{D}_\beta + \bar{D}^\beta D^\alpha\{D_\alpha, \bar{D}_\beta\} - \bar{D}^\beta\{D^\alpha, \bar{D}_\beta\}D_\alpha \\
&+ \ \{\bar{D}^\beta, D^\alpha\}D_\beta\bar{D}_\alpha)\mathcal{J} = 2(\{D^\alpha, \bar{D}^\beta\}\bar{D}_\beta D_\alpha)\mathcal{J}
\end{aligned} \tag{1.35}$$

On the other hand, we also clearly have

$$\partial^{\alpha\beta}(D_\alpha\bar{D}_\beta + D_\beta\bar{D}_\alpha)\mathcal{J} \ \sim \ \{D^\alpha, \bar{D}^\beta\}(D_\alpha\bar{D}_\beta + D_\beta\bar{D}_\alpha)\mathcal{J} = (D^\alpha\bar{D}^\beta D_\alpha\bar{D}_\beta + \bar{D}^\beta D^\alpha D_\alpha\bar{D}_\beta$$

$$
\begin{aligned}
&+ \quad D^\alpha \bar{D}^\beta D_\beta \bar{D}_\alpha + \bar{D}^\beta D^\alpha D_\beta \bar{D}_\alpha)\mathcal{J} = (D^\alpha\{\bar{D}^\beta, D_\alpha\}\bar{D}_\beta + [\bar{D}^\beta, D^\alpha D_\alpha]\bar{D}_\beta \\
&+ \quad D^\alpha\{\bar{D}^\beta, D_\beta\}\bar{D}_\alpha + [\bar{D}^\beta, D^\alpha D_\beta]\bar{D}_\alpha)\mathcal{J} \\
&= \quad (D^\alpha\{\bar{D}^\beta, D_\alpha\}\bar{D}_\beta + \{\bar{D}^\beta, D^\alpha\}D_\alpha\bar{D}_\beta - D^\alpha\{\bar{D}^\beta, D_\alpha\}\bar{D}_\beta \\
&+ \quad \{\bar{D}^\beta, D^\alpha\}D_\beta\bar{D}_\alpha)\mathcal{J} = 2(\{\bar{D}^\beta, D^\alpha\}D_\alpha\bar{D}_\beta)\mathcal{J} \qquad (1.36)
\end{aligned}
$$

Therefore, we see that $\partial^\mu j_\mu = 0$ since

$$
\partial^{\alpha\beta}[D_\alpha, \bar{D}_\beta]\mathcal{J} \sim \partial^{\alpha\beta}\partial_{\theta^\alpha}\partial_{\bar{\theta}^\beta}\mathcal{J} + \cdots = 0 \ , \qquad (1.37)
$$

where the ellipses contain higher-order terms in the Grassman coordinates.

• We can now consider various SUSY invariant Lagrangians. The vector multiplet kinetic terms become

$$
\mathcal{L}_{vec} = -\frac{1}{g^2}\int d^2\theta d^2\bar{\theta}\Sigma^2 = \frac{1}{g^2}\left(\frac{1}{2}D^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\lambda\gamma^\mu\partial_\mu\bar{\lambda}\right) \ . \qquad (1.38)
$$

• For a free massless chiral multiplet, the Lagragian takes the form

$$
\mathcal{L}_{\rm kin} = -\int d^4\theta\bar{\Phi}\Phi = |F|^2 - \partial_\mu\bar{\phi}\partial^\mu\phi + i\psi\gamma^\mu\partial_\mu\bar{\psi} \ . \qquad (1.39)
$$

Note that $\mathcal{J} = \bar{\Phi}\Phi$ is a conserved current multiplet

$$
\begin{aligned}
\bar{D}^2(\bar{\Phi}\Phi) &= \bar{D}^2\bar{\Phi}\Phi \sim \bar{F}\Phi + \cdots = 0 \ , \\
D^2(\bar{\Phi}\Phi) &= \bar{\Phi}D^2\Phi \sim \bar{\Phi}F + \cdots = 0 \ . \qquad (1.40)
\end{aligned}
$$

In the last line, we have used the EOM of F (since the ellipses are higher-order in the Grassmann coordinates, they also vanish by EOM... This is useful for you to check explicitly!). **Exercise:** Prove this is the current multiplet for the $U(1)$ flavor symmetry under which $\Phi \to e^{i\alpha}\Phi$ (and $\bar{\Phi} \to e^{-i\alpha}\bar{\Phi}$).

• We gauge this symmetry by introducing a vector multiplet

$$
\mathcal{L} = -\int d^4\theta\bar{\Phi}e^{2qV}\Phi = -\int d^4\theta(\bar{\Phi}\Phi - 2qV\bar{\Phi}\Phi + 2q^2V^2\bar{\Phi}\Phi) = |F|^2 - D_\mu\bar{\phi}D^\mu\phi + i\psi\gamma^\mu D_\mu\bar{\psi} \ . \qquad (1.41)
$$

The term linear in V is the usual linear gauging of a global symmetry (i.e., the completion of $A^\mu j_\mu$).

• We get 3D $\mathcal{N} = 2$ SQED with N_f flavors if we take the Lagrangian to be

$$
\mathcal{L}_{\rm SQED} = \mathcal{L}_{\rm vec} + \mathcal{L}_{\rm matter} \ , \qquad (1.42)
$$

where

$$
\begin{aligned}
\mathcal{L}_{\mathrm{SQED}} &= -\int d^4\theta \sum_{i=1}^{N_f}(\bar{q}^i e^{2V} q_i + \bar{\tilde{q}}^i e^{-2V}\tilde{q}_i) - \frac{1}{g^2}\int d^2\theta d^2\bar{\theta}\Sigma^2 = \frac{1}{g^2}\Big(\frac{1}{2}D^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\
&- \partial^\mu\sigma\partial_\mu\sigma + i\lambda\gamma^\mu\partial_\mu\bar{\lambda}\Big) + \sum_i(|F_i|^2 - D_\mu\bar{\rho}^i D^\mu\rho_i + i\psi_i\gamma^\mu D_\mu\bar{\psi}^i + |\tilde{F}_i|^2 - D_\mu\bar{\tilde{\rho}}^i D^\mu\tilde{\rho}_i \\
&+ i\tilde{\psi}_i\gamma^\mu D_\mu\bar{\tilde{\psi}}^i - \sigma^2(|\rho_i|^2 + |\tilde{\rho}_i|^2) - D(|\rho_i|^2 - |\tilde{\rho}_i|^2) - i\sigma(\psi_i\bar{\psi}^i - \tilde{\psi}_i\bar{\tilde{\psi}}^i \\
&- \sqrt{2}i(\lambda\psi_i\bar{\rho}^i - \lambda\tilde{\psi}_i\bar{\tilde{\rho}}^i) - \sqrt{2}i(\bar{\lambda}\bar{\psi}^i\rho_i - \bar{\lambda}\bar{\tilde{\psi}}^i\tilde{\rho}_i)) \ . \qquad (1.43)
\end{aligned}
$$

The flavor symmetry here is $U(1)^2 \times SU(N_f) \times SU(N_f)$ (recall: this symmetry commutes with SUSY and the gauge symmetry)... Where one of the $U(1)$ factors is the topological symmetry.

• Let us now better understand the symmetries of the free massless chiral multiplet

$$
\mathcal{L} = -\int d^4\theta\bar{\Phi}\Phi = |F|^2 - \partial_\mu\bar{\phi}\partial^\mu\phi + i\psi\gamma^\mu\partial_\mu\bar{\psi} \ . \qquad (1.44)
$$

We already saw that we have a $U(1)$ flavor symmetry under which $\Phi \to e^{-i\alpha}\Phi$ (and $\bar{\Phi} \to e^{-i\alpha}\bar{\Phi}$... so all components transform in the same way under supersymmetry... can see this from the structure of the $\mathcal{J}$ superfield). In particular, we have

$$
\begin{aligned}
Q_\alpha &\to Q_\alpha \ , \quad [\mathcal{J}, Q_\alpha] = 0 \ , \quad \mathcal{J}(Q_\alpha) = 0 \ , \\
\phi &\to e^{-i\alpha}\phi \ , \quad [\mathcal{J}, \phi] = \phi \ , \quad \mathcal{J}(\phi) = +1 \ , \\
\psi &\to e^{-i\alpha}\psi \ , \quad [\mathcal{J}, \psi_\alpha] = \psi_\alpha \ , \quad \mathcal{J}(\psi_\alpha) = +1 \ , \\
F &\to e^{-i\alpha}F \ , \quad [\mathcal{J}, F] = \psi_\alpha \ , \quad \mathcal{J}(F) = +1 \ . \qquad (1.45)
\end{aligned}
$$

We also have a $U(1)_R$ symmetry as well: under this symmetry

$$
\begin{aligned}
Q_\alpha &\to e^{i\beta}Q_\alpha \ , \quad [R, Q_\alpha] = -Q_\alpha \ , \quad R(Q_\alpha) = -1 \ , \\
\phi &\to e^{-i\frac{\beta}{2}}\phi \ , \quad [R, \phi] = \frac{1}{2}\phi \ , \quad R(\phi) = \frac{1}{2} \ , \\
\psi_\alpha &\to e^{i\frac{\beta}{2}}\psi_\alpha \ , \quad [R, \psi_\alpha] = -\frac{1}{2}\psi_\alpha \ , \quad R(\psi_\alpha) = -\frac{1}{2} \ , \\
F &\to e^{i\frac{3\beta}{2}}F \ , \quad [R, F] = \frac{3}{2}F \ , \quad R(F) = -\frac{3}{2} \ . \qquad (1.46)
\end{aligned}
$$

• What other symmetries do we have? Well, we have superpoincaré: $Q_\alpha, \bar{Q}_\alpha, P_\mu, L_{\mu\nu}$... We have also seen the higher spin symmetries in lecture 1... However, we have another set of symmetries as well....

• For example, we can dilate spacetime as follows $x^\mu \to e^\lambda x^\mu$, where $\lambda \in \mathbb{R}$ (note that e^λ is not a phase). We say x^μ has scaling dimension -1. Then, we get

$$\begin{aligned} S &= \int d^3x \int d^4\theta \bar{\Phi}\Phi = \int d^3x \left(|F|^2 - \partial_\mu \bar{\phi} \partial^\mu \phi + i\psi\gamma^\mu \partial_\mu \bar{\psi}\right) \\ &\to \int d^3x e^{3\lambda} (|F|^2 - e^{-2\lambda} \partial_\mu \bar{\phi} \partial^\mu \phi + e^{-\lambda} \psi \gamma^\mu \partial_\mu \bar{\psi}) \end{aligned} \tag{1.47}$$

To get a symmetry we should see if we can assign scaling transformations to the fields in order to make S invariant... How can we do this? Well, we can clearly take

$$\phi \to e^{-\frac{\lambda}{2}}\phi \ , \quad \bar{\phi} \to e^{-\frac{\lambda}{2}}\bar{\phi} \ , \quad \psi_\alpha \to e^{-\lambda}\psi_\alpha \ , \quad \bar{\psi}_\alpha \to e^{-\lambda}\bar{\psi}_\alpha \ , \quad F \to e^{-\frac{3\lambda}{2}} F \ , \quad \bar{F} \to e^{-\frac{3\lambda}{2}} \bar{F} \ . \tag{1.48}$$

This gives us a scaling symmetry since now S doesn't depend on λ... We say $\Delta(\phi) = \frac{1}{2}$, $\Delta(\psi) = 1$, and $\Delta(F) = \frac{3}{2}$. Since the theory is free, this is also true in the quantum theory. Note that we could have derived the above also from the superspace integral by noting that $d\theta \to e^{-\frac{\lambda}{2}} d\theta$.

• Note that

$$R(\phi) = \frac{1}{2} = \Delta(\phi) \ . \tag{1.49}$$

As we will see next week, this relation is not a conicidence.

• Moreover, the above scaling symmetry is part of a larger group of symmetries: the superconformal group. We will discuss aspects of the above R symmetry (called the superconformal R-symmetry) next week. We will also discuss breaking of this symmetry.