1. Lecture 7:

• In our last lecture, we learned that the Witten index of the theory defined by

$$\mathcal{L} = \int d\theta d\theta^* \left(-\frac{1}{2} g_{IJ}(\Phi) \mathcal{D}\Phi^I \mathcal{D}^\dagger \Phi^J \right) \tag{1.1}$$

is equal to the Euler characteristic of the target space manifold $\mathcal{M}$ (where, recall that $\Phi^I : t \to \mathcal{M}$), i.e.

$$\chi(\mathcal{M}) = \sum_p (-1)^p \mathrm{dim}(H^p) = I_W \ . \tag{1.2}$$

• We then learned about Berry's phase and showed that

$$\begin{aligned} \gamma_n(t) &= \oint_{\mathcal{C}} i\langle n_a(r_i(t))|\partial_{r_i} n_a(r_i(t))dr_i = i\int_S \left(\vec{\nabla} \times \langle n(r_i(t))|\vec{\nabla} n(r_i(t))\rangle\right) \cdot d\vec{S} \\ &= \int_S \left(\sum_{m\neq n} \frac{\langle n|\vec{\nabla} H|m\rangle \times \langle m|\vec{\nabla} H|n\rangle}{(E_n - E_m)^2} \right) \cdot d\vec{S} \ . \end{aligned} \tag{1.3}$$

• For the case of a spin in a magnetic field, $H = \vec{B} \cdot \vec{\sigma}$, you showed on the homework that

$$\gamma_n(t) = \int_S \frac{\vec{B}}{2B^3} \cdot d\vec{S} = 2\pi \ . \tag{1.4}$$

This is the field for a magnetic monopole.

• We then moved on to $\mathcal{N} = (2,2)$ SUSY, which will teach us more about Berry's phase / background fields and will connect to the higher dimensional theories we will study more directly.

• The relevant algebra is (again neglecting central terms, as we did in the $\mathcal{N} = 2$ case)

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0 \ , \quad \{Q_\alpha, \bar{Q}_\beta\} = -2H\epsilon_{\alpha\beta} \ . \tag{1.5}$$

where $\alpha, \beta = \pm$ are fundamental labels of an $SU(2)$ group called $SU(2)_R$ (i.e., Q and $\bar{Q}$ transform separately as doublets—or **2**'s—of $SU(2)_R$) We also have

$$\begin{aligned} \epsilon_{\alpha\beta} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad \sigma^1_{\alpha\beta} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} , \quad \sigma^2_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \\ \sigma^3_{\alpha\beta} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \end{aligned} \tag{1.6}$$

Here $\epsilon_{\alpha\beta}$ and $\epsilon^{\alpha\beta}$ are $SU(2)_R$-invariant tensors that are used to raise and lower spin 1/2 indices (**Exercise:** prove these tensors are invariant).

• The automorphism group is now $SU(2)_R \times U(1)_R$ instead of $U(1)_R$ as before. The quantum numbers of the objects appearing above are

$$H \sim \mathbf{0}_0 \ , \quad Q_\alpha \sim \mathbf{2}_{-1} \ , \quad \bar{Q}_\alpha \sim \mathbf{2}_{+1} \ , \tag{1.7}$$

where the bold letters are the $SU(2)_R$ representations (or dimensions) and the subscripts are the $U(1)_R$ charges.... The corresponding Grassmann coordinates are (e.g., see [1])

$$\theta_\alpha \ , \quad \bar{\theta}^\alpha \equiv (\theta_\alpha)^* \ . \tag{1.8}$$

We have

$$\begin{aligned} \theta_\alpha &= \epsilon_{\alpha\beta}\theta^\beta \ , \quad \theta^\alpha = \epsilon^{\alpha\beta}\theta_\beta \ , \\ \bar{\theta}_\alpha &= \epsilon_{\beta\alpha}\bar{\theta}^\beta \ , \quad \bar{\theta}^\alpha = \epsilon^{\beta\alpha}\bar{\theta}_\beta \ . \end{aligned} \tag{1.9}$$

As you showed on the HW, these coordinates satisfy

$$\begin{aligned} \theta_\alpha\theta_\beta &= \frac{1}{2}\epsilon_{\alpha\beta}\theta^2 \ , \quad \bar{\theta}_\alpha\bar{\theta}_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\bar{\theta}^2 \ , \\ \theta^\alpha\theta^\beta &= -\frac{1}{2}\epsilon^{\alpha\beta}\theta^2 \ , \quad \bar{\theta}^\alpha\bar{\theta}^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\bar{\theta}^2 \ , \end{aligned} \tag{1.10}$$

where we define

$$\theta^2 \equiv \theta^\alpha\theta_\alpha = \epsilon^{\alpha\beta}\theta_\beta\theta_\alpha \ , \quad \bar{\theta}^2 \equiv \bar{\theta}_\alpha\bar{\theta}^\alpha = \epsilon^{\alpha\beta}\bar{\theta}_\beta\bar{\theta}_\alpha \ , \quad \theta\bar{\theta} = \theta_\alpha\bar{\theta}^\alpha = \theta^\alpha\bar{\theta}_\alpha \ . \tag{1.11}$$

In particular, we are using the $SU(2)_R$-invariant $\epsilon^{\alpha\beta}$ tensor to raise indices and the $SU(2)_R$-invariant (inverse) $\epsilon_{\alpha\beta}$ tensor to lower indices (but we should be careful when we use this tensor since we use it and its transpose to act spinors and their conjugates respectively)...

• We define

$$(\eta_\alpha\xi_\beta)^* = \bar{\xi}^\beta\bar{\eta}^\alpha \ , \tag{1.12}$$

and so **Exercise:** Prove that (1.12) implies

$$\partial^*_{\theta^\alpha} = -\partial_{\bar{\theta}_\alpha} \ , \quad \partial^*_{\theta_\alpha} = -\partial_{\bar{\theta}^\alpha} \ . \tag{1.13}$$

Also check that

$$\epsilon^{\alpha\beta}\partial_{\theta^\beta} = -\partial_{\theta_\alpha} \ , \quad \epsilon^{\beta\alpha}\partial_{\bar{\theta}^\beta} = -\partial_{\bar{\theta}_\alpha} \ , \tag{1.14}$$

and

$$(\sigma^a\sigma^b)_{\alpha\beta} = \sigma^a_{\alpha\gamma}\epsilon^{\gamma\delta}\sigma^b_{\delta\beta} = \delta^{ab}\epsilon_{\alpha\beta} + i\epsilon^{abc}\sigma^c_{\alpha\beta} \ , \ (\sigma^a_{\alpha\beta})^* = \epsilon^{\alpha\gamma}\epsilon^{\delta\beta}\sigma^a_{\gamma\delta} \equiv \sigma^{a\alpha\beta} \ , \ \mathrm{Tr}(\sigma^a\sigma^b) = 2\delta^{ab} \ . \tag{1.15}$$

• The resulting SUSY covariant derivatives are (these are consistent with (1.13) and (1.14))

$$\begin{aligned} \mathcal{D}_\alpha &= \partial_{\theta^\alpha} - i\bar{\theta}_\alpha\partial_t \ , \quad \bar{\mathcal{D}}_\alpha = \partial_{\bar{\theta}^\alpha} - i\theta_\alpha\partial_t \ , \\ \mathcal{Q}_\alpha &= \partial_{\theta^\alpha} + i\bar{\theta}_\alpha\partial_t \ , \quad \bar{\mathcal{Q}}_\alpha = \partial_{\bar{\theta}^\alpha} + i\theta_\alpha\partial_t \ . \end{aligned} \tag{1.16}$$

They satisfy the algebra

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta\} = 2i\epsilon_{\alpha\beta}\partial_t \ , \quad \{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_\beta\} = -2i\epsilon_{\alpha\beta}\partial_t \ , \tag{1.17}$$

with all other anti-commutators vanishing, i.e., $\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{\bar{\mathcal{D}}_\alpha, \bar{\mathcal{D}}_\beta\} = \{\mathcal{D}_\alpha, \mathcal{Q}_\beta\} = \{\bar{\mathcal{D}}_\alpha, \mathcal{Q}_\beta\} = \{\bar{\mathcal{D}}_\alpha, \bar{\mathcal{Q}}_\beta\} = \{\mathcal{D}_\alpha, \bar{\mathcal{Q}}_\beta\} = \{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = \left\{\bar{\mathcal{Q}}_\alpha, \bar{\mathcal{Q}}_\beta\right\} = 0$.

Exercise: Prove (1.17).

• To understand the Dirac monopole Berry's phase in the context of $\mathcal{N} = 2(2,2)$ SUSY, we should introduce gauge multiplets: the SUSY completion of gauge fields. Let's consider a $U(1)$ gauged quantum mechanics with some number of chiral multiplet matter fields.

• There are three important representations we will need. The first is the chiral multiplet (which is the generalization of the chiral multiplet we studied in the $\mathcal{N} = 2$ QM case), which is a function of $y = t - i\theta^\alpha\bar{\theta}_\alpha$ (note that $\bar{D}_\alpha y = \bar{D}_\alpha\theta = 0$)...

$$\begin{aligned} \Phi &= \phi(y) + \sqrt{2}\theta^\alpha\psi_\alpha(y) + \theta^2F(y) \\ &= \phi - i\theta^\alpha\bar{\theta}_\alpha\phi' + \frac{1}{4}\theta^2\bar{\theta}^2\phi'' + \sqrt{2}\theta\psi - \frac{i}{\sqrt{2}}\theta^2\bar{\theta}_\alpha\psi'^\alpha + \theta^2F \ . \end{aligned} \tag{1.18}$$

It clearly satisfies $\bar{\mathcal{D}}_\alpha\Phi = 0$. Using the Liebnitz rule, we see that $\bar{\mathcal{D}}_\alpha(\Phi_1\Phi_2) = 0$ if the Φ_i are chiral. Similarly, $\bar{\mathcal{D}}_\alpha(\Phi_1 + \Phi_2) = 0$. Therefore, the primaries again form an object called the "chiral ring." They satisfy $[\bar{Q}_\alpha, \phi] = 0$. Note that we can also prove that the Φ_i form a ring by multiplying Φ_1 and Φ_2 together and using the θ expansion:

$$\begin{aligned} \Phi_1\Phi_2 &= (\phi_1(y) + \sqrt{2}\theta^\alpha\psi_{1\alpha}(y) + \theta^2F_1(y))(\phi_2(y) + \sqrt{2}\theta^\alpha\psi_{2\alpha}(y) + \theta^2F_2(y)) \\ &= \phi_1\phi_2 + \sqrt{2}\theta^\alpha(\phi_1\psi_2 + \phi_2\psi_1) + \theta^2\left(\phi_1F_2 + \phi_2F_1 - \psi_1\psi_2\right) \ . \end{aligned} \tag{1.19}$$

• We also have the anti-chiral field (which is a function of $\bar{y} = t + i\theta^\alpha\bar{\theta}_\alpha$ such that $D_\alpha\bar{y} = D_\alpha\bar{\theta} = 0$)

$$\bar{\Phi} = \bar{\phi}(\bar{y}) - \sqrt{2}\bar{\theta}_\alpha\bar{\psi}^\alpha(\bar{y}) - \bar{\theta}^2\bar{F}(\bar{y})$$

$$= \quad \bar{\phi} + i\theta^\alpha \bar{\theta}_\alpha \bar{\phi}' + \frac{1}{4}\theta^2\bar{\theta}^2\bar{\phi}'' - \sqrt{2}\bar{\theta}\bar{\psi} + \frac{i}{\sqrt{2}}\bar{\theta}^2\theta_\alpha\bar{\psi}'^\alpha - \bar{\theta}^2\bar{F} \ . \tag{1.20}$$

Note that

$$\bar{D}^\alpha \sigma^a_{\alpha\beta} D^\beta \Phi = \bar{D}^\alpha \sigma^a_{\alpha\beta} D^\beta \bar{\Phi} = 0 \ , \tag{1.21}$$

This fact will be crucial when discussing "field strength" superfields (although note that the vector itself is non-propagating).

• The second important multiplet is the (abelian) vector multiplet

$$V = -\theta\bar{\theta}A_0 - \bar{\theta}\sigma^a\theta x_a + i\theta^2 \cdot \bar{\theta}\bar{\lambda} - i\bar{\theta}^2\theta\lambda + \frac{1}{4}\theta^2\bar{\theta}^2 D \ , \tag{1.22}$$

where, as we will see in more detail later, supersymmetric gauge transformations allow us to remove the lower-order terms.... This is because under gauge transformations

$$V \to V + \frac{i}{2}(\Lambda - \bar{\Lambda}) \ , \tag{1.23}$$

where Λ is a chiral superfield. The remaining fields besides the gauge fields are singlets: they transform in the adjoint of $U(1)$...

• Finally, we can use the vector to build a (real) linear multiplet (a generalization of the real multiplet we studied in the $\mathcal{N}=2$ QM case)... In higher dimensions, it will include the gauge field strength...

$$\begin{aligned} \Sigma^a = \frac{1}{2}\bar{D}^\alpha \sigma^a_{\alpha\beta} D^\beta V \ &= \ -x^a + i\theta\sigma^a\bar{\lambda} + i\bar{\theta}\sigma^a\lambda - \bar{\theta}\sigma^a\theta D + \epsilon^a{}_{bc}\bar{\theta}\sigma^b\theta x'^c + \frac{1}{2}\bar{\theta}^2 \cdot \theta\sigma^a\lambda' \\ &- \ \frac{1}{2}\theta^2 \cdot \bar{\theta}\sigma^a\bar{\lambda}' + \frac{1}{4}\theta^2\bar{\theta}^2 x''^a \ , \end{aligned} \tag{1.24}$$

where this multiplet satisfies (note that $\bar{\theta}\sigma^a\bar{\lambda}' = \bar{\theta}_\alpha \sigma^{a\alpha}{}_\beta \bar{\lambda}'^\beta$ and $\theta\sigma^a\lambda' = \theta^\alpha \sigma^{a\beta}_\alpha \lambda'_\beta$)

$$D^2\Sigma^a = \bar{D}^2\Sigma^a = 0 \ . \tag{1.25}$$

The second equality is trivial, and the first follows from the SUSY algebra...

• Finally, note that (1.21) implies that Σ^a is invariant under the gauge transformation described in (1.23) (this is what we expect since the adjoint representation of $U(1)$ is the singlet)...

Exercise: Check that (1.22), (1.24), and (1.20) form representations of the SUSY algebra in (1.17). Also, prove (1.25) and (1.21).

• We now want to construct SUSY-invariant interactions... As in the $\mathcal{N}=2$ case, we can compute SUSY variations as follows

$$\delta\chi = [\eta^\alpha Q_\alpha + \bar\eta^\alpha \bar Q_\alpha, \chi] = \eta^\alpha\left(\partial_{\theta^\alpha} + i\bar\theta_\alpha\partial_t\right)\chi + \bar\eta^\alpha\left(\partial_{\bar\theta^\alpha}\chi + i\theta_\alpha\partial_t\right)\chi \ , \tag{1.26}$$

for any superfield, χ. Note that the highest possible component in any superfield is $\theta^2\bar\theta^2$. We can only get such terms from the ∂_t parts of the supercharges.... Therefore, the top component will transform as a total derivative under SUSY.... So (note, $d^4\theta = d\theta^2 d\bar\theta^2 = d\theta^\alpha d\theta_\alpha d\bar\theta_{\dot\alpha} d\bar\theta^{\dot\alpha}$)

$$\int d^4\theta\chi \ , \tag{1.27}$$

is SUSY invariant.

• As a result, we have the following SUSY invariant Lagrangian (note, $d^4\theta = d\theta^2 d\bar\theta^2 = d\theta^\alpha d\theta_\alpha d\bar\theta_{\dot\alpha} d\bar\theta^{\dot\alpha}$) for the abelian vector multiplet kinetic terms

$$\begin{aligned}\mathcal{L}_{\rm vec} &= \frac{1}{3g^2}\int d^4\theta\Sigma^a\Sigma_a = \frac{1}{3g^2}\left(-\frac{1}{2}x^a x''_a + x'^a x'_a + \frac{3i}{2}\bar\lambda\lambda' - \frac{3i}{2}\bar\lambda'\lambda + \frac{3}{2}D^2\right)\\ &= \frac{1}{g^2}\left(\frac{1}{2}x'^a x'_a + i\bar\lambda\lambda' + \frac{1}{2}D^2\right) \ ,\end{aligned} \tag{1.28}$$

where g^{-2} is, in non-relativistic quantum mechanics, a mass term moving in the x^a direction, but, in accordance with conventions we will see in higher dimensions, can also be thought of as a gauge coupling. Note that we can also add

$$\mathcal{L}_{\rm FI} = \zeta\int d\bar\theta\sigma^a d\theta\Sigma_a = -\zeta D \ . \tag{1.29}$$

This is gauge and SUSY invariant... It is called an "FI" term...

• We have the following SUSY-invariant chiral multiplet kinetic terms (with couplings to the vector multiplet)

$$\begin{aligned}\mathcal{L}_{\rm chiral} &= -\int d^2\theta d^2\bar\theta\bar\Phi e^{2qV}\Phi = |D_t\phi|^2 + \frac{i}{2}\psi D_t\bar\psi - \frac{i}{2}D_t\psi\bar\psi + |F|^2 - q^2x^a x_a|\phi|^2\\ &+ \ qx_a\psi\sigma^a\bar\psi - \sqrt{2}qi\bar\phi\lambda\psi - \sqrt{2}qi\phi\bar\lambda\bar\psi - \frac{q}{2}D|\phi|^2 \ .\end{aligned} \tag{1.30}$$

where

$$\begin{aligned}D_t\phi &= (\partial_t - iqA_0)\phi \ , \quad D_t\bar\phi = (\partial_t + iqA_0)\bar\phi \ ,\\ D_t\psi_\alpha &= (\partial_t - iqA_0)\psi_\alpha \ , \quad D_t\bar\psi_\alpha = (\partial_t + iqA_0)\bar\psi_\alpha \ ,\end{aligned} \tag{1.31}$$

Note that the terms linear in A_0 couple to a current for the $U(1)_F$ symmetry under which

$$\Phi \to \Phi e^{-iq\Lambda} \ , \tag{1.32}$$

for constant $\Lambda \in \mathbb{R}$.... The gauge field, A_0, is around to maintain gauge invariance when we promote Λ to a function of t... In fact, can promote Λ to a chiral superfield and allow for supergauge transformations by requiring (1.23)...

• We are now ready to discuss Berry's phase in SQM.

• As you proved in your homework the curvature of the Berry connection we get from a spin-1/2 particle in a magnetic field with

$$H = \vec{B} \cdot \vec{\sigma} \ , \tag{1.33}$$

has curvature

$$\vec{V} = \vec{\nabla} \times \vec{A} = \frac{\vec{B}}{2B^3} \ . \tag{1.34}$$

where this is computed in the $|\uparrow\rangle$ state.

• In the context of SUSY QM, the Berry connection is a gauge connection over the space of parameters. As we will see, these can be thought of as background fields living in different supermultiplets: these are fields whose values have fixed and are not fluctuating (we will sometimes refer to these parameters / fields as spurions since we will often allow them to transform under corresponding broken (or "spurious") symmetries in order to get selection rules).

• Let us now focus on the case of a single free chiral multiplet, Φ. We will turn off gauge interactions. The reason we do this is intuitively clear: in (1.33) $\vec{B}$ is a background field (it is a set of classical parameters that an experimenter can tune—it is not quantum mechanical on its own, unlike the spin-1/2 matter).

• To understand this statement, let us note that the only terms we can turn on and still have a free theory are mass terms. None-the-less, we will see that certain mass terms act as "background" gauge fields!

• But our goal is to make contact with the spin half example in the intro... We needed three parameters there (three magnetic field components)... In the case of a free chiral, we also have a *real* (as opposed to holomorphic—see below for what this latter mass term looks like) mass.... It is a triplet of $SU(2)_R$, so it has 3 components... Also, it enters the Lagrangian like a vector multiplet...

• Indeed, we can think of a real mass parameter living in a background linear multiplet as in (1.24) that weakly gauges a $U(1)_F$ flavor symmetry... In particular, consider a triplet of mass parameters, m^a (setting them non-zero breaks $SU(2)_R \times U(1)_R \to U(1)'_R \times U(1)_R$, but we can imagine allowing them to transform in order to get selection rules). We have

$$\mathcal{L}_{\text{mass}} = -\int d^4\theta \bar{\Phi} e^{-2\bar{\theta}\sigma^a\theta m_a}\Phi = |\phi'|^2 + i\psi\bar{\psi}' - m^a m_a |\phi|^2 + m^a \psi \sigma_a \bar{\psi} \ , \tag{1.35}$$

where m^a plays the role of the x^a $SU(2)_R$ triplet (note that we have solved the EOM of F and set $F=0$)... *Compare to* (1.30). Essentially, we have fixed $x^a = m^a$ and set the other fields in the multiplet to zero.

• Performing the usual transformation back to the Hamiltonian formulation gives us

$$H = |\pi|^2 + m^a m_a |\phi|^2 + \bar{\psi} m^a \sigma_a \psi \ , \tag{1.36}$$

where

$$[\phi, \pi] = i \ , \quad \pi = -i\partial_\phi \ , \quad \{\psi_\alpha, \bar{\psi}^\beta\} = \delta_\alpha^\beta \ . \tag{1.37}$$

The Fermionic operators can be arranged in creation and annihilation operators and yield the space

$$|0\rangle \ , \quad \bar{\psi}_+|0\rangle \ , \quad \bar{\psi}_-|0\rangle \ , \quad \bar{\psi}_+\bar{\psi}_-|0\rangle \ , \tag{1.38}$$

where

$$\psi_\pm|0\rangle = 0 \ . \tag{1.39}$$

On the space $\{|0\rangle, \bar{\psi}_+\bar{\psi}_-|0\rangle\}$, $H_\psi = \bar{\psi} m^a \sigma_a \psi$ vanishes (**Exercise:** check this). However, on the $\{\bar{\psi}_-|0\rangle, \bar{\psi}_+|0\rangle\}$ subspace we have

$$H_\psi \bar{\psi}_\alpha |0\rangle = m^a \sigma_{a\alpha}^{\ \ \beta} \bar{\psi}_\beta |0\rangle \ . \tag{1.40}$$

• This is just the Hamiltonian we encountered for a spin in a magnetic field on the HW. Therefore, we have that (depending on the state in question)

$$\vec{V} = \pm \frac{\vec{m}}{2m^3} \tag{1.41}$$

This also dovetails nicely with our discussion of fermion bilinears as corresponding to spin half particles in a magnetic field from the beginning of the course.... If you'd like to see more details on the above, have a look at the original paper [2] (although some of the conventions are different).

• Before continuing to QFT, there is one more thing I'd like to cover. Since we have chiral superfields, much as in the $\mathcal{N}=2$ case, we can consider holomorphic and anti-holomorphic SUSY invaraints

$$\int d^2\theta W(\Phi) + \text{h.c.} \ . \tag{1.42}$$

From definition of superfield variation

$$\delta\chi = [\eta^\alpha Q_\alpha + \bar{\eta}^\alpha \bar{Q}_\alpha, \chi] = \eta^\alpha \left(\partial_{\theta^\alpha} + i\bar{\theta}_\alpha \partial_t\right)\chi + \bar{\eta}^\alpha \left(\partial_{\bar{\theta}^\alpha}\chi + i\theta_\alpha \partial_t\right)\chi \ , \tag{1.43}$$

and the expansion in (1.20), we see that $[\eta^\alpha Q_\alpha, F] = 0$. Moreover, $[\bar{\eta}^\alpha \bar{Q}_\alpha, F] \sim \bar{\eta}^\alpha \psi'_\alpha$. This is called the superpotential (note that it is holomorphic).

• Therefore, we can also consider holomorphic mass terms.

$$W = \mu\Phi^2 \ , \tag{1.44}$$

we can think of it as constituting a background chiral multiplet (a chiral multiplet with some fixed field value).

• The general Lagrangian then looks like

$$\begin{aligned} \mathcal{L} &= -\int d^4\theta \varphi^\dagger e^{-2\bar{\theta}\sigma_a\theta m^a}\varphi + \int d^2\theta \mu\varphi^2 - \int d^2\bar{\theta}\bar{\mu}\varphi^2 \\ &= |\phi'|^2 + i\psi\psi'^\dagger - m^2|\phi|^2 + m^a\psi\sigma_a\psi^\dagger - 4|\mu|^2|\phi|^2 + \mu\psi^2 - \bar{\mu}\psi^{\dagger 2} \ , \end{aligned} \tag{1.45}$$

where $m = \sqrt{m^a m_a}$.

• On the homework, you will analyze the case $m = 0$, $\mu \neq 0$. We have already analyzed $\mu = 0$, $m \neq 0$. Also, you will study the case $m, \mu \neq 0$. Somewhat surprisingly, this latter case turns out not to have a SUSY vacuum! You will show this explicitly by constructing the Hamiltonian and checking there are no SUSY groundstates (you should do this by looking at the groundstates of the bosonic and fermionic Hamiltonians and summing up the resulting energies).

• However, there is a more conceptual proof: thinking of μ and m^a as comprising background fields yields a term in the bosonic potential of the form

$$V \supset 4m^2|\mu|^2 > 0 \ , \tag{1.46}$$

where the factor of four is due to the charge of μ needed to make $\mu\Phi^2$ invariant under the corresponding $U(1)$ (here we say that μ is a spurion for the symmetry under which Φ rotates by a phase). We will see that this statement is related to a discussion we will have in higher dimensions once we understand the Higgs mechanism: it will be related to a statement about the absence of certain types of "mixed branch" vacua. We will get to this soon—after discussing the (SUSY) Higgs mechanism.

• We have been working in 0+1 dimensions so far. Now, let's move up to 2+1 dimensions (I will just call it 3D... we will occasionally go between Lorentzian and Euclidean signatures... Have $SO(2,1)$ and $SO(3)$... Anti-symm matrices in $SO(3)$ case $X^T X = \mathbb{1}$ and expand $X = e^{\epsilon^a T_a}$... In Lorentzian case, get factor of $\eta^{\mu\nu}$ instead in the definition). Note that momentum must appear because Lorentz transformations relate H to P.

• For the rest of the module we will mostly be concerned with the 3D $\mathcal{N}=2$ algebra

$$\{Q_\alpha, \bar{Q}_\beta\} = 2\gamma^\mu_{\alpha\beta} P_\mu \ , \quad \gamma^\mu_{\alpha\beta} = \gamma^\mu_{\beta\alpha} \ , \quad \mu = 1,2,3 \ , \tag{1.47}$$

Comment: This algebra looks quite similar to the $\mathcal{N} = (2,2)$ SQM algebra, and we will see why, but note there are also a few differences: here α, β are spacetime spinor indices (as opposed to internal R-symmetry indices; note that in both cases, the symmetries in question do not commute with the supercharges)... Also, there is no longer just the Hamiltonian sitting on the RHS of (1.47). Instead, special relativity in 3D forces us to include momentum generators in the spatial directions as well. Also, note that the gamma matrices are symmetric, i.e., we have spin 1 or vector generators.

• There is now a $U(1)_R$ automorphism ($SU(2)_R$ is no longer present, it is replaced by a spacetime symmetry)... We will come back again and again to the important role played by $U(1)_R$ in the coming lectures.

• An aside on spinors: spinors are in the double cover of the space-time symmetry group. If we are in Euclidean space, then this is the double cover of $SO(3)$, i.e., $SU(2) = \mathrm{Spin}(3)$. If we are in Lorentzian space, then this is the double cover of $SO(2,1)$, i.e., $SL(2,\mathbb{R}) = \mathrm{Spin}(2,1)$. We will spend much of the remainder of this module in Lorentzian signature and take

$$\gamma^\mu_{\alpha\beta} = \left(\sigma^0, \sigma^1, \sigma^3\right) \ . \tag{1.48}$$

where

$$\sigma^0_{\alpha\beta} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \quad \sigma^1_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma^3_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{1.49}$$

The following generators generate $SL(2,R)$

$$\epsilon^{\beta\gamma}\sigma^0_{\alpha\gamma}=\sigma^{0\beta}_\alpha = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}~,\quad \epsilon^{\beta\gamma}\sigma^1_{\alpha\gamma}=\sigma^{1\beta}_\alpha=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}~,\quad \epsilon^{\beta\gamma}\sigma^3_{\alpha\gamma}=\sigma^{3\beta}_\alpha=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}~. \quad (1.50)$$

Note that

$$(\gamma^\mu)_\alpha^\beta(\gamma^\nu)_\beta^\lambda=\eta^{\mu\nu}\delta_\alpha^\lambda+\epsilon^{\mu\nu\rho}(\gamma_\rho)_\alpha^\lambda~. \quad (1.51)$$

The main difference is that $SO(2,1)$ has real 2-component spinors while $SU(2)$ does not... The supercharges form a complex 2 component spinor anyway, so can use either space-time symmetry group. See Polchinski volume II for a discussion of spinors in various dimensions...

• Both this 3D $\mathcal{N}=2$ algebra and the $\mathcal{N}=(2,2)$ SQM algebra can be obtained via dimensional reduction of the 4D $\mathcal{N}=1$ SUSY algebra.

• What is dimensional reduction? It is a process to start from some quantum system in d space-time dimensions and reduce it to a quantum system in $d-r<d$ space-time dimensions.

• Suppose these r dimensions form some compact manifold, $\mathcal{M}_r$ (e.g., $\mathcal{M}_r=T^r=S^1\times\cdots\times S^1$). Suppose $\mathcal{M}_r$ has some characteristic length-scale, L (this could be the period of the circles in the T^r). Then, quantum mechanics tells us that $p=n_i/L$ for $n_i\in\mathbb{Z}$ (and $i=1,\cdots,r$). As we take $L\to0$, $p\to\infty$ and so too the energy... Therefore, in this limit, the only finite energy configurations are those that are independent of the extra dimensions... These have $n_i=0$... Specializing to these modes that have no dependence on $\mathcal{M}_r$, we get the dimensional reduction. This is equivalent to setting momentum to zero in the internal dimensions....

• Note that symmetries like rotations of these internal dimensions become internal symmetries of the dimensionally reduced theory (will see this below).

• Let us return to 4D→3D SUSY. There are unfortunately many conventions at play here... To get to 3d $\mathcal{N}=2$, we start in 4D from the Wess and Bagger conventions

$$\{Q_\alpha,\bar{Q}_{\dot\alpha}\}=2\sigma^\mu_{\alpha\dot\alpha}P_\mu~, \quad (1.52)$$

$$\begin{aligned}\sigma^0_{\alpha\dot\beta} &= -\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}~,\quad \sigma^1_{\alpha\dot\beta}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}~,\quad \sigma^2_{\alpha\dot\beta}=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}~,\\ \sigma^3_{\alpha\dot\beta} &= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}~. \end{aligned} \quad (1.53)$$

we have here both "dotted" (e.g., "$\dot{\alpha}$") and "undotted" (e.g., "α") spinor indices here. This is because $\text{Spin}(3,1) = SL(2,\mathbb{C})$, so twice as many types of spinor reps. Similarly, for the Euclidean case, $\text{Spin}(4) = SU(2) \times SU(2)$, so there are twice as many types of spinor reps.

• Let us now reduce by, say, setting to zero momentum in the x^2 direction of $\mathbb{R}^4 \to \mathbb{R}^3 \times S^1$ (where x^2 parameterizes the S^1). This means, we set $P_2 = 0$ and obtain the algebra we had for 3D $\mathcal{N} = 2$ (we drop the dotted index because in 3D there is no distinction between dotted and undotted).

• To get to our $\mathcal{N} = (2,2)$ SQM, we would start from a slightly different convention for our $\sigma^\mu_{\alpha\dot{\alpha}}$, e.g.,

$$\sigma^0_{\alpha\dot{\alpha}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma^1_{\alpha\dot{\alpha}} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \ , \quad \sigma^2_{\alpha\dot{\alpha}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \quad \sigma^3_{\alpha\dot{\alpha}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{1.54}$$

We would then set $P^3 = P^2 = P^1 = 0$. The spin group of the transverse 3D become generators of the $SU(2)_R$ symmetry of the quantum mechanics...

• Back to QFT: roughly, we should treat free quantum fields as operator valued functions of space obeying equal time commutation relations (in the Heisenberg picture)

$$[\varphi_a(x_i,t), \varphi_b(y_i,t)] = [\pi^a(x_i,t), \pi^b(y_i,t)] = 0 \ , \quad [\pi^b(x_i,t), \phi_a(y_i,t)] = -i\delta^{(2)}(x_i - y_i)\delta_a^b \ . \tag{1.55}$$

• Since we are studying objects that depend on both space and time, we should modify our superspace differential operators. They become

$$\mathcal{Q}_\alpha = \partial_{\theta^\alpha} + i\gamma^{\mu\beta}_\alpha \bar{\theta}_\beta \partial_\mu \ , \quad \bar{\mathcal{Q}}_\alpha = -\partial_{\bar{\theta}^\alpha} - i\gamma^{\mu\beta}_\alpha \theta_\beta \partial_\mu \ . \tag{1.56}$$

where we raise and lower with $\epsilon^{\alpha\beta}$ and $\epsilon_{\alpha\beta}$ as follows

$$\epsilon^{\alpha\beta}\theta_\beta = \theta^\alpha \ , \quad \epsilon^{\alpha\beta}\bar{\theta}_\beta = \bar{\theta}^\alpha \ , \quad \epsilon_{\alpha\beta}\theta^\beta = \theta_\alpha \ , \quad \epsilon_{\alpha\beta}\bar{\theta}^\beta = \bar{\theta}_\alpha \ . \tag{1.57}$$

The SUSY covariant derivatives are now

$$\mathcal{D}_\alpha = \partial_{\theta^\alpha} - i\gamma^{\mu\beta}_\alpha \bar{\theta}_\beta \partial_\mu \ , \quad \bar{\mathcal{D}}_\alpha = -\partial_{\bar{\theta}^\alpha} + i\gamma^{\mu\beta}_\alpha \theta_\beta \partial_\mu \ . \tag{1.58}$$

These quantities satisfy

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta\} = -2i\gamma^\mu_{\alpha\beta}\partial_\mu \ , \quad \{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_\beta\} = 2i\gamma^\mu_{\alpha\beta}\partial_\mu \ . \tag{1.59}$$

Other useful identities include **(Exercise!)**

$$\mathcal{D}^\alpha \bar{\mathcal{D}}_\alpha = \bar{\mathcal{D}}^\alpha \mathcal{D}_\alpha \ , \quad \{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \cdots = 0 \ . \tag{1.60}$$

We also have the SUSY integration definitions

$$\int d^2\theta\theta^2 = 1 \ , \quad \int d^2\bar{\theta}\bar{\theta}^2 = -1 \ , \quad \int d^4\theta\theta^2\bar{\theta}^2 = -1 \ . \tag{1.61}$$

• What are some representations of the 3D $\mathcal{N}=2$ SUSY algebra? Well, we again have our friend the chiral multiplet, which is a function of θ_α and $y^\mu = x^\mu - i\theta\gamma^\mu\bar{\theta}$

$$\begin{aligned} \Phi &= \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y) = \phi(x) - i\theta\gamma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{1}{4}\theta^2\bar{\theta}^2\partial^2\phi(x) + \sqrt{2}\theta\psi(x) \\ &+ \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi(x)\gamma^\mu\bar{\theta} + \theta^2 F(x) \ . \end{aligned} \tag{1.62}$$

Similarly, we have an anti-chiral multiplet

$$\begin{aligned} \bar{\Phi} &= \bar{\phi}(y) - \sqrt{2}\bar{\theta}\bar{\psi}(y) - \bar{\theta}^2\bar{F}(y) = \bar{\phi}(x) + i\theta\gamma^\mu\bar{\theta}\partial_\mu\bar{\phi}(x) - \frac{1}{4}\theta^2\bar{\theta}^2\partial^2\bar{\phi}(x) - \sqrt{2}\bar{\theta}\bar{\psi}(x) \\ &- \frac{i}{\sqrt{2}}\bar{\theta}^2\theta\gamma^\mu\partial_\mu\bar{\psi}(x) + \bar{\theta}^2\bar{F}(x) \ . \end{aligned} \tag{1.63}$$

Note that these multiplets satisfy $0 = \mathcal{D}^\alpha\bar{\mathcal{D}}_\alpha\Phi = \bar{\mathcal{D}}^\alpha\mathcal{D}_\alpha\Phi$, and similarly for $\bar{\Phi}$.

• Next week we will continue with our exploration of 3D $\mathcal{N}=2$.

References

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