1. Lecture 6: $\mathcal{N} = (2, 2)$ QM and Berry Phase

• In our last lecture, we learned how to write the 1D particle in superspace

$$\mathcal{L} = \int d\theta d\theta^* \left(-\frac{1}{2}\mathcal{D}\Phi\mathcal{D}^\dagger\Phi + W(\Phi) \right) . \tag{1.1}$$

In this lecture, we will generalize this discussion in a way that makes the connection between the Witten Index and topological quantities clearer.

• To get to this point, let us first recall that we showed there is a one-to-one correspondence

$$\text{SUSYgd.states} \leftrightarrow \ker(Q^\dagger)/\mathrm{Im}(Q^\dagger) \ , \tag{1.2}$$

where $\ker(Q^\dagger)$ is made up of $Q^\dagger$-closed SUSY states (i.e., those annihilated by $Q^\dagger$). States that are $Q^\dagger$ of something else are in $\mathrm{Im}(Q^\dagger) \subset \mathrm{Ker}(Q^\dagger)$. The "/" means that we work modulo terms in $\mathrm{Im}(Q^\dagger)$ (i.e., if two states in $\ker(Q^\dagger)$ differ by such terms, we identify these two states). Since $Q^{\dagger 2} = 0$, this defines some notion of cohomology....

• **Comment 1:** Replace $Q^\dagger \to d$ (where $d : \Omega^r(\mathcal{M}) \to \Omega^{r+1}(\mathcal{M})$), and you will have the cohomology you were studying last semester in the Differential Geometry module (that cohomology was called de Rham cohomology... we call the closed r forms modulo the exact ones the "rth de Rham cohomology group", $H^r(\mathcal{M})$). This is the same idea in a different guise... In fact, we will see precisely that $Q^\dagger \to d$ in some interesting examples in a moment... Here $\mathcal{M}$ is a "manifold." This is a space that locally looks like $\mathbb{R}^N$ (if it is N-dimensional), but globally has some more non-trivial structure (that you patch together via transition functions, e.g., as in the case of a sphere). Don't worry too much about technical details of what a manifold is.

• **Comment 2:** The Euler characteristic is a useful topological invariant of $\mathcal{M}$. It is defined as

$$\chi = \sum(-1)^p b_p = \sum(-1)^p \dim(H^r) \ . \tag{1.3}$$

Should remind you a bit of the Witten index... This is no accident. Now, it is useful (for visualization purposes) to note that χ is related to the alternating sum of dimensions of homology groups by de Rham's theorem.

• Roughly speaking, the p^{th} homology group, H_p, is the set of p-dimensional cycles modulo boundaries. Here a cycle, $\mathcal{C} \subset \mathcal{M}$ is a closed submanifold (i.e., $\mathcal{C}$ is compact without boundary, i.e., $\partial\mathcal{C} = 0$). Boundaries here are cycles that are themselves boundaries of another submanifold. There is then a natural pairing between elements of H_p and H^p via integration. **Give homework example and circle, which has $\chi = 0$ (related to I... not an accident). Give torus example ($\chi = 0$).**

• More generally for a 2D surface, $\chi = 2 - 2g$, where g is the genus (a.k.a., the number of handles). Momentarily, we will relate it to I_W. Same will hold in higher dimensions.

• Before continuing, let us also clear something up from HW2: when ϕ is compact, we don't need to care about $\phi \to \pm\infty$.... Can still have more normalizable solutions than in the non-compact case (since fields don't die off)... Just need to check that wavefunctions obey boundary conditions...

• **Example (the "nlσm"[1]):** Let us use this discussion on a more sophisticated example, consider

$$\mathcal{S} = \int dt d\theta d\theta^* \Big(-\frac{1}{2} g_{IJ}(\Phi) \mathcal{D}\Phi^I \mathcal{D}^\dagger \Phi^J \Big) \tag{1.4}$$

Where $I, J = 1, \cdots N$, $\Phi^I : t \to M$, and M is an N-dimensional compact manifold (again, don't worry about the precise definition of this here: it is a space that locally looks like $\mathbb{R}^N$ but that has to be "smoothly pieced together" as in the example, say, of a sphere)...

• Here g_{IJ} is a metric on M (it is Riemannian in order for energies to be bounded from below), and the derivatives are the SUSY covariant derivatives we introduced above...

• We will also assume M is compact (as in the case of a finite radius sphere; a sufficient condition for this is that every Cauchy sequence of points—i.e., every sequence of points where distances between points go to zero— in M converges in M and that the diameter of the manifold—i.e., the supremum of geodesic distances— is finite).

• Expanding out in components, we find

$$\mathcal{L} = \frac{1}{2} g_{IJ} \phi'^I \phi'^J + \frac{i}{2} g_{IJ} \psi^{*I} \frac{D}{Dt} \psi^J + \frac{1}{8} R_{IJKL} \psi^{*I} \psi^J \psi^{*K} \psi^L \ , \tag{1.5}$$

where R_{IJKL} is the Riemann tensor of M, and, if we interpret $\psi^{*I} \sim d\phi^I \in \Omega^1(M)$ as a 1-form (with ψ^I as a tangent vector) then D/Dt is the usual covariant derivative.

Exercise: Check the above statement (note that the first term should not involve derivatives of the metric by what we have said above, while the second term involves both zero and

[1]This strange terminology dates to ur-QCD physics and the fact that this model is similar in nature to a model for a particular resonance Murray Gell-Mann was studying: the σ.

one derivative terms since it is the usual covariant derivative... the last term must involve two derivatives of the metric, which is consistent with the fact that the Riemann tensor has two derivatives...)

• The supercharges generalize what we wrote before (for zero superpotential)[2]

$$Q = i\sum_I \psi^I \pi_I \ , \quad Q^\dagger = -i\sum_I \psi^{*I}\pi_I \ , \quad \pi^I = -i\frac{D}{D\phi_I} \ , \quad H = \Delta \tag{1.7}$$

where the second to last expression is the covariant derivative on M and H is the Laplacian....

• Now, let's build the quantum states of this theory. We start from the anti-commutation relations (Clifford algebra)

$$\{\psi^{*I}, \psi^J\} = g^{IJ} \ . \tag{1.8}$$

So the theory is built on the state $|\Omega\rangle$ that is annihilated by all ψ^I. Acting with raising operators gives us

$$F_{I_1,\cdots I_p}(\phi)\psi^{*I_1}\cdots\psi^{*I_p}|\Omega\rangle \ . \tag{1.9}$$

These clearly correspond to p-forms, and, in this case

$$Q^\dagger = d : \Omega^p \to \Omega^{p+1} \ . \tag{1.10}$$

Moreover,

$$Q = d^* \equiv *d* : \Omega^p \to \Omega^{p-1} \ , \tag{1.11}$$

where $*$ is the Hodge star.[3]

Exercise Check the above discussion.

• Then,

$$2H = dd^* + d^*d = 2\Delta \ , \tag{1.12}$$

where Δ is the Laplacian on M. As we saw above, SUSY groundstates in one to one correspondence with $\ker Q^\dagger/\mathrm{Im} Q^\dagger$. On the other hand, using our identification $Q^\dagger = d$, we see the number of SUSY groundstates with fermion number p (assuming $|\Omega\rangle$ has fermion

[2]Recall that

$$Q = \frac{1}{\sqrt{\hbar}}\psi(W' + i\pi) \ , \quad Q^\dagger = \frac{1}{\sqrt{\hbar}}\psi^\dagger(W' - i\pi) \ , \quad H = \frac{1}{2}(\pi^2 + W'^2 - [\psi^\dagger, \psi]W'') \tag{1.6}$$

[3]Taking $\omega = \frac{1}{r!}\omega_{\mu_1\cdots\mu_r} d\psi^{*\mu_1}\cdots d\psi^{*\mu_r}$, we have $*\omega = \frac{\sqrt{\det(g)}}{r!(N-r)!}\omega_{\mu_1\cdots\mu_r}\epsilon^{\mu_1\cdots\mu_r}{}_{\nu_{r+1}\cdots\nu_N} d\psi^{*\nu_{r+1}}\cdots d\psi^{*\nu_N}$...

number zero) is just given by $b_p = \mathrm{dim} H^p(M)$ since the states in (1.9) are closed p forms... The Witten index is then the Euler character

$$\mathrm{Tr}(-1)^F = \sum_p (-1)^p b_p = \chi \ . \tag{1.13}$$

In the case of the theory on the circle, we see: $I = 1 - 1 = 0$

• Now I want to change topics and discuss SQM with more SUSY... This will lead us to an interesting discussion of Berry's phase / Geometrical phase in SQM (applications of this to SUSY QFT in higher dimensions remain an area of research). We will learn some advanced concepts that usually appear in QFT courses but here in QM.

• The basic idea behind Berry's phase is to take a quantum system with some parameters $r_i \in \mathcal{M}$. Let the corresponding Hamiltonian have a spectrum with quantum numbers n_a, $|n_a(r_i(t))\rangle$. One then varies the parameters adiabatically (i.e., slowly) and (for our purposes) assumes no level crossing. Then, under such evolution, the eigenstates are expected to remain eigenstates and can be followed along the corresponding path (i.e., we start in a particular eigenstate and stay in that eigenstate). When we come back to the state we started from along a closed loop, $\mathcal{C}$, then the original state comes back to itself up to a phase that depends only on the geometry of $\mathcal{M}$ and the topology of $\mathcal{C}$ (this phase can be non-abelian).

• Ansatz

$$|\Psi(t, r_i)\rangle = \exp\left(i\gamma_n(t)\right) \exp\left(-\frac{i}{\hbar}\int_0^t E_n(r_i(t))\right) |n_a(r_i(t))\rangle \ . \tag{1.14}$$

Plugging into the Schrödinger equation

$$H(r_i(t))|\Psi(t, r_i)\rangle = i\hbar\partial_t |n_a(r_i(t))\rangle \ , \tag{1.15}$$

and sandwiching with $\langle n_a(r_i(t))|$ yields

$$\frac{d}{dt}\gamma_n(t) = i\langle n_a(r_i(t))|\partial_{r_i} n_a(r_i(t))\rangle \frac{dr_i}{dt} \ , \tag{1.16}$$

and so

$$\gamma_n(t) = \oint_{\mathcal{C}} i\langle n_a(r_i(t))|\partial_{r_i} n_a(r_i(t))dr_i \ . \tag{1.17}$$

Also can have non-abelian generalization.

• **Comment:** Note that Stoke's theorem is useful here... Indeed,

$$\gamma_n(t) = \oint_{\mathcal{C}} i\langle n_a(r_i(t))|\partial_{r_i} n_a(r_i(t))dr_i = i\int_S \left(\vec{\nabla} \times \langle n(r_i(t))|\vec{\nabla} n(r_i(t))\rangle\right) \cdot d\vec{S} \ . \tag{1.18}$$

RHS is clearly gauge invariant... However, non-trivial sometimes to evaluate RHS... So, defining $\vec{V} \equiv i\vec{\nabla} \times \langle n|\vec{\nabla} n\rangle$, we have

$$\vec{V} = i\langle\vec{\nabla} n| \times |\vec{\nabla} n\rangle = i\sum_m \langle\vec{\nabla} n|m\rangle \times \langle m|\vec{\nabla} n\rangle \tag{1.19}$$

These latter quantities can be related to matrix elements of $\vec{\nabla} H$. To see this, note that

$$\vec{\nabla}\langle m|H|n\rangle = E_n\vec{\nabla}\langle m|n\rangle = 0 = \langle\vec{\nabla} m|H|n\rangle + \langle m|\vec{\nabla} H|n\rangle + \langle m|H|\vec{\nabla} n\rangle \tag{1.20}$$

Note also that orthonormalization of the basis implies

$$\vec{\nabla}\langle m|n\rangle = \vec{\nabla}\delta_{mn} = 0 = \langle\vec{\nabla} m|n\rangle + \langle m|\vec{\nabla} n\rangle \ . \tag{1.21}$$

Plugging this into the previous equation yields

$$0 = (E_n - E_m)\langle\vec{\nabla} m|n\rangle + \langle m|\nabla H|n\rangle \ . \tag{1.22}$$

Note, by (1.21), we have that the term in (1.19) with $m = n$ vanishes since it is the cross product of a vector with itself. Therefore, we obtain

$$\vec{V} = \sum_{m\neq n} \frac{\langle n|\vec{\nabla} H|m\rangle \times \langle m|\vec{\nabla} H|n\rangle}{(E_n - E_m)^2} \ , \tag{1.23}$$

with γ the corresponding surface integral.

Exercise: Spin 1/2 particle in magnetic field, $\vec{B} \in \mathbb{R}^3$ has $H = \vec{B} \cdot \vec{\sigma}$... Here the $\vec{B}$ is the set of parameters, i.e., $B_i = r_i$ for $i = 1, 2, 3$. So, we have $\vec{\nabla} H = \vec{\sigma}$. Then, given above, check that

$$\vec{\nabla} \times \vec{A} = \frac{\vec{B}}{2B^3} \ . \tag{1.24}$$

This is the connection of a Dirac monopole... We will find a SUSY construction of this monopole soon... It will illustrate an important idea in SUSY: background fields.

• We now move to $\mathcal{N} = (2, 2)$ SUSY. It is more complicated than the $\mathcal{N} = 2$ algebra we considered before... Two reasons we want to study this algebra: it will allow us to get an interesting SUSY version of Berry's phase AND will connect directly to the 2+1D discussion we will have soon via dimensional reduction. The algebra is (again neglecting central terms, as we did in the $\mathcal{N} = 2$ case)

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0 \ , \quad \{Q_\alpha, \bar{Q}_\beta\} = -2H\epsilon_{\alpha\beta} \ . \tag{1.25}$$

where $\alpha, \beta = \pm$ are fundamental labels of an $SU(2)$ group called $SU(2)_R$ (i.e., Q and $\bar{Q}$ transform separately as doublets—or $\mathbf{2}$'s—of $SU(2)_R$) $\sigma^a_{\alpha\beta}$ are the generators of this $SU(2)_R$ and transform as a triplet (or $\mathbf{3}$ of $SU(2)_R$—the index $a = 1, 2, 3$ is a triplet index... We are summing over raised and lowered indices... We raise and lower a with δ^{ab} and δ_{ab} respectively)... We also have

$$
\begin{aligned}
\epsilon_{\alpha\beta} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad \sigma^1_{\alpha\beta} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} , \quad \sigma^2_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \\
\sigma^3_{\alpha\beta} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .
\end{aligned}
\tag{1.26}
$$

Here $\epsilon_{\alpha\beta}$ and $\epsilon^{\alpha\beta}$ are $SU(2)_R$-invariant tensors that are used to raise and lower spin 1/2 indices (**Exercise:** prove these tensors are invariant).

The automorphism group is now

$$
SU(2)_R \times U(1)_R \tag{1.27}
$$

instead of $U(1)_R$ as before. The quantum numbers of the objects appearing above are

$$
H \sim \mathbf{0}_0 \ , \quad Q_\alpha \sim \mathbf{2}_{-1} \ , \quad \bar{Q}_\alpha \sim \mathbf{2}_{+1} \ , \tag{1.28}
$$

where the bold letters are the $SU(2)_R$ representations (or dimensions) and the subscripts are the $U(1)_R$ charges.... The corresponding Grassmann coordinates are (e.g., see [1])

$$
\theta_\alpha \ , \quad \bar{\theta}^\alpha \equiv (\theta_\alpha)^* \ . \tag{1.29}
$$

We have

$$
\begin{aligned}
\theta_\alpha &= \epsilon_{\alpha\beta}\theta^\beta \ , \quad \theta^\alpha = \epsilon^{\alpha\beta}\theta_\beta \ , \\
\bar{\theta}_\alpha &= \epsilon_{\beta\alpha}\bar{\theta}^\beta \ , \quad \bar{\theta}^\alpha = \epsilon^{\beta\alpha}\bar{\theta}_\beta \ .
\end{aligned}
\tag{1.30}
$$

These coordinates satisfy **(Exercise)**

$$
\begin{aligned}
\theta_\alpha\theta_\beta &= \frac{1}{2}\epsilon_{\alpha\beta}\theta^2 \ , \quad \bar{\theta}_\alpha\bar{\theta}_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\bar{\theta}^2 \ , \\
\theta^\alpha\theta^\beta &= -\frac{1}{2}\epsilon^{\alpha\beta}\theta^2 \ , \quad \bar{\theta}^\alpha\bar{\theta}^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\bar{\theta}^2 \ ,
\end{aligned}
\tag{1.31}
$$

where we define

$$
\theta^2 \equiv \theta^\alpha\theta_\alpha = \epsilon^{\alpha\beta}\theta_\beta\theta_\alpha \ , \quad \bar{\theta}^2 \equiv \bar{\theta}_\alpha\bar{\theta}^\alpha = \epsilon^{\alpha\beta}\bar{\theta}_\beta\bar{\theta}_\alpha \ , \quad \theta\bar{\theta} = \theta_\alpha\bar{\theta}^\alpha = \theta^\alpha\bar{\theta}_\alpha \ . \tag{1.32}
$$

In particular, we are using the $SU(2)_R$-invariant $\epsilon^{\alpha\beta}$ tensor to raise indices and the $SU(2)_R$-invariant (inverse) $\epsilon_{\alpha\beta}$ tensor to lower indices (but we should be careful when we use this tensor since we use it and its transpose to act spinors and their conjugates respectively)...

• We define

$$(\eta_\alpha \xi_\beta)^* = \bar{\xi}^\beta \bar{\eta}^\alpha \ , \tag{1.33}$$

and so **Exercise:** Prove that (1.33) implies

$$\partial^*_{\theta^\alpha} = -\partial_{\bar{\theta}_\alpha} \ , \quad \partial^*_{\theta_\alpha} = -\partial_{\bar{\theta}^\alpha} \ . \tag{1.34}$$

Also check that

$$\epsilon^{\alpha\beta}\partial_{\theta^\beta} = -\partial_{\theta_\alpha} \ , \quad \epsilon^{\beta\alpha}\partial_{\bar{\theta}^\beta} = -\partial_{\bar{\theta}_\alpha} \ , \tag{1.35}$$

and

$$(\sigma^a\sigma^b)_{\alpha\beta} = \sigma^a_{\alpha\gamma}\epsilon^{\gamma\delta}\sigma^b_{\delta\beta} = \delta^{ab}\epsilon_{\alpha\beta} + i\epsilon^{abc}\sigma^c_{\alpha\beta} \ , \ (\sigma^a_{\alpha\beta})^* = \epsilon^{\alpha\gamma}\epsilon^{\delta\beta}\sigma^a_{\gamma\delta} \equiv \sigma^{a\alpha\beta} \ , \ \mathrm{Tr}(\sigma^a\sigma^b) = 2\delta^{ab} \ . \tag{1.36}$$

• The resulting SUSY covariant derivatives are (these are consistent with (1.34) and (1.35))

$$\begin{aligned} \mathcal{D}_\alpha &= \partial_{\theta^\alpha} - i\bar{\theta}_\alpha\partial_t \ , \quad \bar{\mathcal{D}}_\alpha = \partial_{\bar{\theta}^\alpha} - i\theta_\alpha\partial_t \ , \\ \mathcal{Q}_\alpha &= \partial_{\theta^\alpha} + i\bar{\theta}_\alpha\partial_t \ , \quad \bar{\mathcal{Q}}_\alpha = \partial_{\bar{\theta}^\alpha} + i\theta_\alpha\partial_t \ . \end{aligned} \tag{1.37}$$

They satisfy the algebra

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta\} = 2i\epsilon_{\alpha\beta}\partial_t \ , \quad \{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_\beta\} = -2i\epsilon_{\alpha\beta}\partial_t \ , \tag{1.38}$$

with all other anti-commutators vanishing, i.e., $\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{\bar{\mathcal{D}}_\alpha, \bar{\mathcal{D}}_\beta\} = \{\mathcal{D}_\alpha, \mathcal{Q}_\beta\} = \{\bar{\mathcal{D}}_\alpha, \mathcal{Q}_\beta\} = \{\bar{\mathcal{D}}_\alpha, \bar{\mathcal{Q}}_\beta\} = \{\mathcal{D}_\alpha, \bar{\mathcal{Q}}_\beta\} = \{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = \{\bar{\mathcal{Q}}_\alpha, \bar{\mathcal{Q}}_\beta\} = 0$.

Exercise: Prove (1.38).

• To understand the Dirac monopole Berry's phase in the context of $\mathcal{N} = 2(2,2)$ SUSY, we should introduce gauge multiplets: the SUSY completion of gauge fields. Let's consider a $U(1)$ gauged quantum mechanics with some number of chiral multiplet matter fields.

• There are three important representations we will need. The first is the chiral multiplet (which is the generalization of the chiral multiplet we studied in the $\mathcal{N} = 2$ QM case), which is a function of $y = t - i\theta^\alpha\bar{\theta}_\alpha$ (note that $\bar{D}_\alpha y = \bar{D}_\alpha\theta = 0$)...

$$\Phi = \phi(y) + \sqrt{2}\theta^\alpha\psi_\alpha(y) + \theta^2 F(y)$$

$$= \phi - i\theta^\alpha\bar\theta_\alpha\phi' + \frac{1}{4}\theta^2\bar\theta^2\phi'' + \sqrt{2}\theta\psi - \frac{i}{\sqrt{2}}\theta^2\bar\theta_\alpha\psi'^\alpha + \theta^2 F \ . \qquad (1.39)$$

It clearly satisfies $\bar{\mathcal{D}}_\alpha\Phi = 0$. Using the Liebnitz rule, we see that $\bar{\mathcal{D}}_\alpha(\Phi_1\Phi_2) = 0$ if the Φ_i are chiral. Similarly, $\bar{\mathcal{D}}_\alpha(\Phi_1 + \Phi_2) = 0$. Therefore, the primaries again form an object called the "chiral ring." They satisfy $[\bar{Q}_\alpha, \phi] = 0$. Note that we can also prove that the Φ_i form a ring by multiplying Φ_1 and Φ_2 together and using the θ expansion:

$$\begin{aligned}\Phi_1\Phi_2 &= (\phi_1(y) + \sqrt{2}\theta^\alpha\psi_{1\alpha}(y) + \theta^2 F_1(y))(\phi_2(y) + \sqrt{2}\theta^\alpha\psi_{2\alpha}(y) + \theta^2 F_2(y)) \\ &= \phi_1\phi_2 + \sqrt{2}\theta^\alpha(\phi_1\psi_2 + \phi_2\psi_1) + \theta^2\left(\phi_1 F_2 + \phi_2 F_1 - \frac{1}{2}\psi_1\psi_2\right) \ . \end{aligned} \qquad (1.40)$$

• We also have the anti-chiral field (which is a function of $\bar{y} = t + i\theta^\alpha\bar\theta_\alpha$ such that $D_\alpha\bar{y} = D_\alpha\bar\theta = 0$)

$$\begin{aligned}\bar\Phi &= \bar\phi(\bar y) - \sqrt{2}\bar\theta_\alpha\bar\psi^\alpha(\bar y) - \bar\theta^2\bar F(\bar y) \\ &= \bar\phi + i\theta^\alpha\bar\theta_\alpha\bar\phi' + \frac{1}{4}\theta^2\bar\theta^2\bar\phi'' - \sqrt{2}\bar\theta\bar\psi + \frac{i}{\sqrt{2}}\bar\theta^2\theta_\alpha\bar\psi'^\alpha - \bar\theta^2\bar F \ . \end{aligned} \qquad (1.41)$$

Note that

$$\bar{D}^\alpha\sigma^a_{\alpha\beta}D^\beta\Phi = \bar{D}^\alpha\sigma^a_{\alpha\beta}D^\beta\bar\Phi = 0 \ , \qquad (1.42)$$

This fact will be crucial when discussing "field strength" superfields (although note that the vector itself is non-propagating).

• The second important multiplet is the (abelian) vector multiplet

$$V = -\theta\bar\theta A_0 - \bar\theta\sigma^a\theta x_a + i\theta^2\cdot\bar\theta\bar\lambda - i\bar\theta^2\theta\lambda + \frac{1}{4}\theta^2\bar\theta^2 D \ , \qquad (1.43)$$

where, as we will see in more detail later, supersymmetric gauge transformations allow us to remove the lower-order terms.... This is because under gauge transformations

$$V \to V + \frac{i}{2}(\Lambda - \bar\Lambda) \ , \qquad (1.44)$$

where Λ is a chiral superfield. The remaining fields besides the gauge fields are singlets: they transform in the adjoint of $U(1)$...

• Finally, we can use the vector to build a (real) linear multiplet (a generalization of the real multiplet we studied in the $\mathcal{N} = 2$ QM case)... In higher dimensions, it will include the gauge field strength...

$$\Sigma^a = \frac{1}{2}\bar{D}^\alpha\sigma^a_{\alpha\beta}D^\beta V \ = \ -x^a + i\theta\sigma^a\bar\lambda + i\bar\theta\sigma^a\lambda - \bar\theta\sigma^a\theta D + \epsilon^a{}_{bc}\bar\theta\sigma^b\theta x'^c + \frac{1}{2}\bar\theta^2\cdot\theta\sigma^a\lambda'$$

$$- \quad \frac{1}{2}\theta^2 \cdot \bar{\theta}\sigma^a\bar{\lambda}' + \frac{1}{4}\theta^2\bar{\theta}^2 x''^a \ , \tag{1.45}$$

where this multiplet satisfies (note that $\bar{\theta}\sigma^a\bar{\lambda}' = \bar{\theta}_\alpha \sigma^{a\alpha}{}_\beta \bar{\lambda}'^\beta$ and $\theta\sigma^a\lambda' = \theta^\alpha \sigma^{a\beta}_\alpha \lambda'_\beta$)

$$D^2\Sigma^a = \bar{D}^2\Sigma^a = 0 \ . \tag{1.46}$$

The second equality is trivial, and the first follows from the SUSY algebra...

• Finally, note that (1.42) implies that Σ^a is invariant under the gauge transformation described in (1.44) (this is what we expect since the adjoint representation of $U(1)$ is the singlet)...

Exercise: Check that (1.43), (1.45), and (1.41) form representations of the SUSY algebra in (1.38). Also, prove (1.46) and (1.42).

• We now want to construct SUSY-invariant interactions... As in the $\mathcal{N} = 2$ case, we can compute SUSY variations as follows

$$\delta\chi = [\eta^\alpha Q_\alpha + \bar{\eta}^\alpha \bar{Q}_\alpha, \chi] = \eta^\alpha \left(\partial_{\theta^\alpha} + i\bar{\theta}_\alpha \partial_t\right) + \bar{\eta}^\alpha \left(\partial_{\bar{\theta}^\alpha}\chi + i\theta_\alpha \partial_t\right)\chi \ , \tag{1.47}$$

for any superfield, χ. Note that the highest possible component in any superfield is $\theta^2\bar{\theta}^2$. We can only get such terms from the ∂_t parts of the supercharges.... Therefore, the top component will transform as a total derivative under SUSY.... So (note, $d^4\theta = d\theta^2 d\bar{\theta}^2 = d\theta^\alpha d\theta_\alpha d\bar{\theta}_{\dot{\alpha}} d\bar{\theta}^{\dot{\alpha}}$)

$$\int d^4\theta\chi \ , \tag{1.48}$$

is SUSY invariant.

• As a result, we have the following SUSY invariant Lagrangian (note, $d^4\theta = d\theta^2 d\bar{\theta}^2 = d\theta^\alpha d\theta_\alpha d\bar{\theta}_{\dot{\alpha}} d\bar{\theta}^{\dot{\alpha}}$) for the abelian vector multiplet kinetic terms

$$\begin{aligned} \mathcal{L}_{\text{vec}} &= \frac{1}{3g^2}\int d^4\theta \Sigma^a \Sigma_a = \frac{1}{3g^2}\left(-\frac{1}{2}x^a x''_a + x'^a x'_a + \frac{3i}{2}\bar{\lambda}\lambda' - \frac{3i}{2}\bar{\lambda}'\lambda + \frac{3}{2}D^2\right) \\ &= \frac{1}{g^2}\left(\frac{1}{2}x'^a x'_a + i\bar{\lambda}\lambda' + \frac{1}{2}D^2\right) \ , \end{aligned} \tag{1.49}$$

where g^{-2} is, in non-relativistic quantum mechanics, a mass term moving in the x^a direction, but, in accordance with conventions we will see in higher dimensions, can also be thought of as a gauge coupling. Note that we can also add

$$\mathcal{L}_{\text{FI}} = \zeta \int d\bar{\theta}\sigma^a d\theta \Sigma_a = -\zeta D \ . \tag{1.50}$$

This is gauge and SUSY invariant... It is called an "FI" term...

• We have the following SUSY-invariant chiral multiplet kinetic terms (with couplings to the vector multiplet)

$$\begin{aligned}\mathcal{L}_{\rm chiral} &= -\int d^2\theta d^2\bar\theta \bar\Phi e^{2qV}\Phi = |D_t\phi|^2 + \frac{i}{2}\psi D_t\bar\psi - \frac{i}{2}D_t\psi\bar\psi + |F|^2 - q^2x^ax_a|\phi|^2 \\ &+ qx_a\psi\sigma^a\bar\psi - \sqrt{2}qi\bar\phi\lambda\psi - \sqrt{2}qi\phi\bar\lambda\bar\psi - \frac{q}{2}D|\phi|^2 \ . \end{aligned} \tag{1.51}$$

where

$$\begin{aligned} D_t\phi &= (\partial_t - iqA_0)\phi \ , \quad D_t\bar\phi = (\partial_t + iqA_0)\bar\phi \ , \\ D_t\psi_\alpha &= (\partial_t - iqA_0)\psi_\alpha \ , \quad D_t\bar\psi_\alpha = (\partial_t + iqA_0)\bar\psi_\alpha \ , \end{aligned} \tag{1.52}$$

Note that the terms linear in A_0 couple to a current for the $U(1)_F$ symmetry under which

$$\Phi \to \Phi e^{-iq\Lambda} \ , \tag{1.53}$$

for constant $\Lambda \in \mathbb{R}$.... The gauge field, A_0, is around to maintain gauge invariance when we promote Λ to a function of t... In fact, can promote Λ to a chiral superfield and allow for supergauge transformations by requiring (1.44)...

• Next week we will use the above discussion to study Berry's phase in the context of $\mathcal{N} = (2,2)$ SUSY. This will end our discussion of SQM.

References

[1] D.-E. Diaconescu & R. Entin, *"A Nonrenormalization theorem for the d = 1, N=8 vector multiplet"*, Phys. Rev. **D56**, 8045 (1997), `hep-th/9706059`