1. Lecture 5:

• In our last lecture, we began the process of taking the 1D particle written as a Lagrangian integrated over (space)-time

$$S = \int dt \left[\frac{1}{2} \left(\frac{d\phi}{dt} \right)^2 + i\psi^* \frac{d\psi}{dt} - \frac{1}{2} W'^2 + \frac{1}{2} \left[\psi^*, \psi \right] W'' \right] \ , \tag{1.1}$$

and writing it as a superspace Lagrangian, $\mathcal{L}_S$, written over superspace

$$S = \int dt d\theta d\theta^* \mathcal{L}_S \ . \tag{1.2}$$

• To get to this point, we found it useful to introduce a real superfield

$$\Phi(t, \theta, \theta^*) = \phi(t) + \theta\psi(t) - \theta^*\psi^*(t) + \theta\theta^* F(t) \ , \tag{1.3}$$

containing all the degrees of freedom above (and also the real F field).

• Given a superfield, it was useful to introduce differential operators for the supercharges (complementing the differential operator, $\mathcal{H} = i\partial_t$ for the Hamiltonian)

$$\mathcal{Q} = \frac{\partial}{\partial\theta} + i\theta^* \frac{\partial}{\partial t} \ , \qquad \mathcal{Q}^\dagger = \frac{\partial}{\partial\theta^*} + i\theta \frac{\partial}{\partial t} \ , \tag{1.4}$$

where we saw that $\{\mathcal{Q}, \mathcal{Q}^\dagger\} = 2\mathcal{H}$ and $\{\mathcal{Q}, \mathcal{Q}\} = \{\mathcal{Q}^\dagger, \mathcal{Q}^\dagger\} = 0$.

• The operators in (1.4) are useful because they allow us to define a simple action of supersymmetry on a superfield (i.e., superfields form a representation of the SUSY algebra with generators given in (1.4))

$$\delta\Phi = [\eta Q + \eta^* Q^\dagger, \Phi] = (\eta\mathcal{Q} + \eta^*\mathcal{Q}^\dagger)\Phi \ . \tag{1.5}$$

Note that

$$\begin{aligned} [\eta Q, \Phi] &= \eta(\partial_\theta + i\theta^*\partial_t)\Phi = \eta\psi - \theta^*\eta(F + i\phi') - \theta\theta^*\eta i\psi' \ , \\ [\eta^* Q^\dagger, \Phi] &= \eta^*(\partial_{\theta^*} + i\theta\partial_t)\Phi = -\eta^*\psi^* + \theta\eta^*(F - i\phi') - \theta\theta^*\eta^* i\psi'^* \ , \end{aligned} \tag{1.6}$$

Therefore, matching the variations term-by-term in the Grassmann expansion in

$$\delta\Phi = \delta\phi + \theta\delta\psi - \theta^*\delta\psi^* + \theta\theta^*\delta F \ , \tag{1.7}$$

we have that $\delta\phi = \eta\psi - \eta^*\psi^*$, $\delta\psi = \eta^*(F + i\phi')$, $\delta\psi^* = \eta(F - i\phi')$, and $\delta F = -i(\eta\psi' + \eta^*\psi'^*)$, which agreed with what we found using canonical commutation relations provided we identify $F = W'(\phi)$ (we will see how this identification occurs shortly).

• Note that the variation of the top component (i.e., the $\mathcal{O}(\theta\theta^*)$ component) is a total derivative. More generally, for any (say, real) superfield this is true since

$$X(t,\theta,\theta^*) = a(t) + \theta b(t) - \theta^* b^*(t) + \theta\theta^* c(t) \ , \tag{1.8}$$

and the only way to get a variation in c is to involve the ∂_t component of $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ (the other components would have to act on terms cubic in Grassmann parameters, which vanish).

• We can now use Grassmann integration to write SUSY invariant actions. Recall

$$\int d\theta = \int d\theta^* = \int d\theta\theta^* = \int d\theta^*\theta = 0 \ , \quad \int d\theta\theta = \int d\theta^*\theta^* = 1 \ , \quad d\theta d\theta^* = -d\theta^* d\theta \ , \tag{1.9}$$

which is equivalent to Grassmann differentiation[1]

$$\{\partial_\theta, \theta\} = \{\partial_{\theta^*}, \theta^*\} = 1 \ , \quad \{\partial_\theta, \theta^*\} = \{\partial_{\theta^*}, \theta\} = 0 \ , \tag{1.10}$$

• Therefore,

$$\int dt d\theta d\theta^* X \tag{1.11}$$

is a SUSY invariant for any X.

• In particular, it is natural to consider $W(\Phi)$ (i.e., the superpotential)

$$W(\Phi) = W(\phi) + \theta W'(\phi)\psi - \theta^* W'(\phi)\psi^* + \theta\theta^*(FW'(\phi) + \frac{1}{2}W''(\phi)[\psi,\psi^*]) \ , \tag{1.12}$$

Therefore, we have that

$$\delta\left(\int dt d\theta d\theta^* W(\Phi)\right) = \delta \int dt F_W = \int dt \partial_t(\cdots) = 0 \ , \tag{1.13}$$

and

$$\begin{aligned} \int dt d\theta d\theta^* W(\Phi) &= \int dt d\theta d\theta^* \theta\theta^* (FW'(\phi) + \frac{1}{2}W''(\phi)[\psi,\psi^*]) \\ &= -FW'(\phi) - \frac{1}{2}W''(\phi)[\psi,\psi^*] = -FW'(\phi) + \frac{1}{2}[\psi^*,\psi]W''(\phi) \ . \end{aligned} \tag{1.14}$$

which reproduces the fermionic potential terms in (1.1). The peculiar $FW'(\phi)$ term will make more sense when we add kinetic terms for ϕ and ψ, and we will see how to reproduce

[1] We should then define $\int d\theta\theta\theta^* = \theta^*$ and $\int d\theta^*\theta\theta^* = -\int d\theta^*\theta^*\theta = -\theta$.

(1.1). These additional terms, when appropriately completed will be SUSY invariant on their own as well.

• As is often the case when we introduce new structures in physics, we need to introduce covariant derivatives to make derivatives transform in a "nice" way under the new structure. In this case, the covariant superderivatives **(which we previewed on the homework)** are

$$\mathcal{D} = \partial_\theta - i\theta^*\partial_t \ , \quad \mathcal{D}^\dagger = \partial_{\theta^*} - i\theta\partial_t \ . \tag{1.15}$$

They differ from the corresponding supercharge differential operators by taking $t \to -t$. We have

$$\{\mathcal{D}, \mathcal{Q}\} = \{\mathcal{D}^\dagger, \mathcal{Q}\} = \{\mathcal{D}, \mathcal{Q}^\dagger\} = \{\mathcal{D}^\dagger, \mathcal{Q}^\dagger\} = \{\mathcal{D}, \mathcal{D}\} = \{\mathcal{D}^\dagger, \mathcal{D}^\dagger\} = 0 \ , \quad \{\mathcal{D}, \mathcal{D}^\dagger\} = -2\mathcal{H} \ . \tag{1.16}$$

and

$$\mathcal{D}\Phi = \psi + \theta^*(F - i\phi') + \theta\theta^* i\psi' \ , \quad \mathcal{D}^\dagger\Phi = -\psi^\dagger - \theta(F + i\phi') + \theta\theta^* i\psi'^\dagger \ . \tag{1.17}$$

Clearly the top component is a total derivative. Therefore, $\int d\theta d\theta^* \mathcal{D}\Phi$ is not a deformation of the Lagrangian (similar statements hold for $\mathcal{D}\Phi \to \mathcal{D}^\dagger\Phi$). Note also, we have that (**Note:** the raison d'etre for covariant derivatives is that $\mathcal{D}\Phi$ should transform under SUSY in the same way as Φ)

$$\delta\mathcal{D}\Phi = [\eta Q + \eta^* Q^\dagger, \mathcal{D}\Phi] = \mathcal{D}[\eta Q + \eta^* Q^\dagger, \Phi] = \mathcal{D}(\eta\mathcal{Q} + \eta^*\mathcal{Q}^\dagger)\Phi = (\eta\mathcal{Q} + \eta^*\mathcal{Q}^\dagger)\mathcal{D}\Phi \ , \tag{1.18}$$

and so superderivatives of superfields transform like superfields under SUSY (in the second equality above, we have used the fact that Q and $Q^\dagger$ act on fields while $\mathcal{D}$ acts on coordinates; in the last equality we have used the anti-commutativity of the superderivatives and supercharges).

• Therefore, we have

$$\mathcal{S} = \int dt d\theta d\theta^* f(\Phi, \mathcal{D}\Phi, \mathcal{D}^\dagger\Phi) \ , \tag{1.19}$$

is supersymmetric for real f (this should be clear since it is a real function of superfields and hence can be written as a sum of real superfields). The most general such Lagrangian with at most two derivatives is then

$$\mathcal{S} = \int dt d\theta d\theta^* \left(-\frac{1}{2}\mathcal{D}\Phi\mathcal{D}^\dagger\Phi + W(\Phi)\right) \ . \tag{1.20}$$

It is instructive to expand this Lagrangian out in coordinates. Doing so, we obtain

$$\begin{aligned}\mathcal{S} &= \int dt\left(-\frac{1}{2}(-i\psi'(-\psi^*)+\psi(-i\psi'^*)-F^2-\phi'^2)-W'F+\frac{1}{2}W''(\phi)[\psi^*,\psi]\right)\\ &= \int dt\left(\frac{1}{2}(\phi'^2+F^2)+i\psi^*\psi'-W'F+\frac{1}{2}W''[\psi^*,\psi]\right)\ .\end{aligned}\tag{1.21}$$

• Note that F does not appear with a derivative: it is an auxiliary field... Its equations of motion can be solved classically (it appears quadratically)... this is called "integrating out the auxiliary field"

$$F = W'(\phi)\ ,\tag{1.22}$$

which derives the identity we used before... Moreover, plugging this result into the above action gives us what we found before in (1.1). **Thus, superspace gives a nice linear realization of SUSY even in interacting theories and also allows us to easily write out SUSY Lagrangians**.... Can go to more fields

$$\mathcal{S} = \int dt d\theta d\theta^*\left(-\frac{1}{2}\sum_i \mathcal{D}\Phi_i\mathcal{D}^\dagger\Phi_i + W(\Phi_i)\right)\tag{1.23}$$

• The Φ multiplet we introduced above is an example of a "long" multiplet of fields (**Note: these are different multiplets than the multiplets of states we have discussed so far...**): unless $F=0$ (e.g., if $W=0$), it has every component of superspace non-zero (this later case is an example of "superconformal quantum mechanics"... $H=\frac{d^2}{dt^2}(\phi^2)$). We can also get short multiplets. These will be useful later. An important example are chiral (anti-chiral) multiplets:

$$\mathcal{D}^\dagger X = 0\ ,\quad \mathcal{D}X^\dagger = 0\tag{1.24}$$

Such chiral superfields are functions of $\tau = t - i\theta\theta^*$ and θ (note that $\mathcal{D}^\dagger\tau = \mathcal{D}^\dagger\theta = 0$) while such anti-chiral superfields are functions of $\tau^* = t + i\theta\theta^*$ and θ^* (note that $\mathcal{D}\tau^* = \mathcal{D}\theta^* = 0$), so

$$\begin{aligned}X &= \chi(\tau)+\theta\psi(\tau) = \chi(t)+\theta\psi(t)-i\theta\theta^*\chi'(t)\ ,\\ X^* &= \chi^*(\tau)-\theta^*\psi^*(\tau) = \chi^*(t)-\theta^*\psi^*(t)+i\theta\theta^*\chi'^*(t)\ .\end{aligned}\tag{1.25}$$

• We can then construct new SUSY invariants by considering terms of the form

$$\delta\mathcal{L} = \int d\theta X + \int d\theta^* X^*\ .\tag{1.26}$$

Some such terms cannot be written as integrals over all of superspace. Indeed, it is easy to check from

$$\delta X = [\eta Q + \eta^* Q^\dagger, X] = (\eta \mathcal{Q} + \eta^* \mathcal{Q}^\dagger) X \ , \tag{1.27}$$

that $[\eta Q, \psi] = 0$ and $[\eta^* Q^\dagger, \psi] = -2i\chi'$ (which is a total derivative) and so the above is indeed an invariant (note $[\eta^* Q^\dagger, \psi^*] = 0$ and $[\eta Q, \psi^*] = 2i\chi'^*$).

• Note also from the above that

$$[Q^\dagger, \chi] = [Q, \chi^\dagger] = 0 \ . \tag{1.28}$$

These are precisely the Q-closed (anti-chiral) / $Q^\dagger$-closed (chiral) operators we encountered in our previous lecture. We learn that they are primaries (i.e., first components) of anti-chiral and chiral superfields.... We get Q-exact operators as primaries of, e.g., $\mathcal{D}\Phi$ (i.e., ψ)and $Q^\dagger$-exact operators as primaries of $\mathcal{D}^\dagger\Phi$ (i.e., ψ^*).

• Easy to show that they form a structure called a ring (known in the SUSY literature as a **"chiral ring"**)... Recall that a ring is a set equipped with addition and multiplication. Addition is associative, commutative, has a 0 element, and an inverse. Multiplication is associative and has a unit element. Finally multiplication and addition are compatible in the sense that multiplication is distributive w.r.t. addition... All of the above conditions are easily verified... It is also simple to see that

$$\mathcal{D}^\dagger X_{1,2} = 0 \Rightarrow \mathcal{D}^\dagger(X_1 + X_2) = \mathcal{D}^\dagger(X_1 X_2) = 0 \ . \tag{1.29}$$

The multiplication identity follows from the fact that $\theta^2 = 0$ (if we just expand in terms of θ and τ). These rings will play an important role in the field theories we analyze later. Needless to say, the above properties can easily be generalized to anti-chiral superfields (note that if X is chiral, then $X^\dagger$ is anti-chiral).

• *Spaces parameterized by higher dimensional analogs of these operators (e.g., moduli spaces and conformal manifolds) will naturally give rise to QM: one reason is that the corresponding operators—like QM operators—do not have singularities when we bring them together...*

• The above quantum mechanical system in (1.20) has a $U(1)_R$-symmetry, i.e., an internal $U(1)$ symmetry that doesn't commute with SUSY

$$[R, Q] = -Q \ , \quad [R, Q^\dagger] = Q^\dagger \ , \quad [R, H] = 0 \ , \tag{1.30}$$

which we define to mean that Q has R-charge -1 and $Q^\dagger$ has R-charge $+1$. It is easy to see that under this symmetry ϕ has R-charge zero, ψ has R-charge -1, and $\psi^\dagger$ has R-charge $+1$ (the auxiliary field, F has R-charge zero). Finally, note that the R-charge is

$$R = \psi^\dagger \psi \ . \tag{1.31}$$

As we will soon see, the existence of such extra symmetries in many SUSY theories will lead to powerful constraints.

• In previous lectures, we saw the Witten index was topological: it didn't depend on explicit length scales, β (i.e., $\frac{d}{d\beta} I_W = 0$)... Now, we want to link the Witten index to topological invariants of manifolds... In order to do this, it will be helpful to formalize the relation we saw above and in the previous lectures between Q and $Q^\dagger$ and cohomology. In particular, let us first show there is a one-to-one correspondence

$$\text{SUSYgd.states} \leftrightarrow \ker(Q^\dagger)/\text{Im}(Q^\dagger) \ , \tag{1.32}$$

where $\ker(Q^\dagger)$ is made up of $Q^\dagger$-closed SUSY states (i.e., those annihilated by $Q^\dagger$). States that are $Q^\dagger$ of something else are in $\text{Im}(Q^\dagger) \subset \text{Ker}(Q^\dagger)$. The "/" means that we work modulo terms in $\text{Im}(Q^\dagger)$ (i.e., if two states in $\ker(Q^\dagger)$ differ by such terms, we identify these two states). Since $Q^{\dagger 2} = 0$, this defines some notion of cohomology....

• To see (1.32), we wish to show $Q^\dagger|\chi\rangle = 0$ implies $|\chi\rangle = Q^\dagger|\psi\rangle$ if and only if χ is not a SUSY ground state... Suppose $|\chi\rangle$ is not a SUSY groundstate. Then, the corresponding $E \neq 0$ and $|\psi\rangle = \frac{1}{2E} Q|\chi\rangle \neq 0$. Acting with $Q^\dagger$ then yields $|\chi\rangle = Q^\dagger|\psi\rangle$. Next, let us suppose $|\chi\rangle$ is a SUSY ground state. In this case, $E = 0$ but $|\chi\rangle \neq Q^\dagger|\psi\rangle$ since otherwise $|\psi\rangle$ would have zero energy and we would have $0 = Q^\dagger|\psi\rangle = |\chi\rangle$. **q.e.d.**

• **Comment 1:** Replace $Q^\dagger \to d$ (where $d : \Omega^r(\mathcal{M}) \to \Omega^{r+1}(\mathcal{M})$), and you will have the cohomology you were studying last semester in the Differential Geometry module (that cohomology was called de Rham cohomology... we call the closed r forms modulo the exact ones the "rth de Rham cohomology group", $H^r(\mathcal{M})$). This is the same idea in a different guise... In fact, we will see precisely that $Q^\dagger \to d$ in some interesting examples in a moment... Here $\mathcal{M}$ is a "manifold." This is a space that locally looks like $\mathbb{R}^N$ (if it is N-dimensional), but globally has some more non-trivial structure (that you patch together via transition functions, e.g., as in the case of a sphere). Don't worry too much about technical details of what a manifold is.

• **Comment 2:** The Euler characteristic is a useful topological invariant of $\mathcal{M}$. It is defined as

$$\chi = \sum(-1)^p b_p = \sum(-1)^p \mathrm{dim}(H^r) \ . \tag{1.33}$$

Should remind you a bit of the Witten index... This is no accident. Now, it is useful (for visualization purposes) to note that χ is related to the alternating sum of dimensions of homology groups by de Rham's theorem.

• Roughly speaking, the p^{th} homology group, H_p, is the set of p-dimensional cycles modulo boundaries. Here a cycle, $\mathcal{C} \subset \mathcal{M}$ is a closed submanifold (i.e., $\mathcal{C}$ is compact without boundary, i.e., $\partial\mathcal{C} = 0$). Boundaries here are cycles that are themselves boundaries of another submanifold. There is then a natural pairing between elements of H_p and H^p via integration. **Give homework example and circle, which has $\chi = 0$ (related to I... not an accident). Give torus example ($\chi = 0$).**

• More generally for a 2D surface, $\chi = 2 - 2g$, where g is the genus (a.k.a., the number of handles). Momentarily, we will relate it to I_W. Same will hold in higher dimensions.

• Before continuing, let us also clear something up from HW2: when ϕ is compact, we don't need to care about $\phi \to \pm\infty$.... Can still have more normalizable solutions than in the non-compact case (since fields don't die off)... Just need to check that wavefunctions obey boundary conditions...

• **Example (the "nlσm"[2]):** Let us use this discussion on a more sophisticated example, consider

$$\mathcal{S} = \int dt d\theta d\theta^* \Big(-\frac{1}{2} g_{IJ}(\Phi)\mathcal{D}\Phi^I\mathcal{D}^\dagger\Phi^J\Big) \tag{1.34}$$

Where $I, J = 1, \cdots N$, $\Phi^I : t \to M$, and M is an N-dimensional compact manifold (again, don't worry about the precise definition of this here: it is a space that locally looks like $\mathbb{R}^N$ but that has to be "smoothly pieced together" as in the example, say, of a sphere)...

• Here g_{IJ} is a metric on M (it is Riemannian in order for energies to be bounded from below), and the derivatives are the SUSY covariant derivatives we introduced above...

• We will also assume M is compact (as in the case of a finite radius sphere; a sufficient condition for this is that every Cauchy sequence of points—i.e., every sequence of points

[2] This strange terminology dates to ur-QCD physics and the fact that this model is similar in nature to a model for a particular resonance Murray Gell-Mann was studying: the σ.

where distances between points go to zero— in M converges in M and that the diameter of the manifold—i.e., the supremum of geodesic distances— is finite).

• Expanding out in components, we find

$$\mathcal{L} = \frac{1}{2} g_{IJ} \phi'^I \phi'^J + \frac{i}{2} g_{IJ} \psi^{*I} \frac{D}{Dt} \psi^J + \frac{1}{8} R_{IJKL} \psi^{*I} \psi^J \psi^{*K} \psi^L \ , \tag{1.35}$$

where R_{IJKL} is the Riemann tensor of M, and, if we interpret $\psi^{*I} \sim d\phi^I \in \Omega^1(M)$ as a 1-form (with ψ^I as a tangent vector) then D/Dt is the usual covariant derivative.

Exercise: Check the above statement (note that the first term should not involve derivatives of the metric by what we have said above, while the second term involves both zero and one derivative terms since it is the usual covariant derivative... the last term must involve two derivatives of the metric, which is consistent with the fact that the Riemann tensor has two derivatives...)

• The supercharges generalize what we wrote before (for zero superpotential)[3]

$$Q = i \sum_I \psi^I \pi_I \ , \quad Q^\dagger = -i \sum_I \psi^{*I} \pi_I \ , \quad \pi^I = -i \frac{D}{D\phi_I} \ , \quad H = \Delta \tag{1.37}$$

where the second to last expression is the covariant derivative on M and H is the Laplacian....

• Now, let's build the quantum states of this theory. We start from the anti-commutation relations (Clifford algebra)

$$\{\psi^{*I}, \psi^J\} = g^{IJ} \ . \tag{1.38}$$

So the theory is built on the state $|\Omega\rangle$ that is annihilated by all ψ^I. Acting with raising operators gives us

$$F_{I_1, \cdots I_p}(\phi) \psi^{*I_1} \cdots \psi^{*I_p} |\Omega\rangle \ . \tag{1.39}$$

These clearly correspond to p-forms, and, in this case

$$Q^\dagger = d : \Omega^p \to \Omega^{p+1} \ . \tag{1.40}$$

Moreover,

$$Q = d^* \equiv *d* : \Omega^p \to \Omega^{p-1} \ , \tag{1.41}$$

[3]Recall that

$$Q = \frac{1}{\sqrt{\hbar}} \psi (W' + i\pi) \ , \quad Q^\dagger = \frac{1}{\sqrt{\hbar}} \psi^\dagger (W' - i\pi) \ , \quad H = \frac{1}{2}(\pi^2 + W'^2 - [\psi^\dagger, \psi] W'') \tag{1.36}$$

where $*$ is the Hodge star.[4]

Exercise Check the above discussion.

• Then,

$$2H = dd^* + d^*d = 2\Delta \ , \tag{1.42}$$

where Δ is the Laplacian on M. As we saw above, SUSY groundstates in one to one correspondence with $\mathrm{ker}Q^\dagger/\mathrm{Im}Q^\dagger$. On the other hand, using our identification $Q^\dagger = d$, we see the number of SUSY groundstates with fermion number p (assuming $|\Omega\rangle$ has fermion number zero) is just given by $b_p = \mathrm{dim}H^p(M)$ since the states in (1.39) are closed p forms... The Witten index is then the Euler character

$$\mathrm{Tr}(-1)^F = \sum_p (-1)^p b_p = \chi \ . \tag{1.43}$$

In the case of the theory on the circle, we see: $I = 1 - 1 = 0$

• Now I want to change topics and discuss SQM with more SUSY... This will lead us to an interesting discussion of Berry's phase / Geometrical phase in SQM (applications of this to SUSY QFT in higher dimensions remain an area of research). We will learn some advanced concepts that usually appear in QFT courses but here in QM.

• The basic idea behind Berry's phase is to take a quantum system with some parameters $r_i \in \mathcal{M}$. Let the corresponding Hamiltonian have a spectrum with quantum numbers n_a, $|n_a(r_i(t))\rangle$. One then varies the parameters adiabatically (i.e., slowly) and (for our purposes) assumes no level crossing. Then, under such evolution, the eigenstates are expected to remain eigenstates and can be followed along the corresponding path (i.e., we start in a particular eigenstate and stay in that eigenstate). When we come back to the state we started from along a closed loop, $\mathcal{C}$, then the original state comes back to itself up to a phase that depends only on the geometry of $\mathcal{M}$ and the topology of $\mathcal{C}$ (this phase can be non-abelian).

• Ansatz

$$|\Psi(t, r_i)\rangle = \exp\left(i\gamma_n(t)\right) \exp\left(-\frac{i}{\hbar}\int_0^t E_n(r_i(t))\right) |n_a(r_i(t))\rangle \ . \tag{1.44}$$

Plugging into the Schrödinger equation

$$H(r_i(t))|\Psi(t, r_i)\rangle = i\hbar\partial_t |n_a(r_i(t))\rangle \ , \tag{1.45}$$

[4]Taking $\omega = \frac{1}{r!}\omega_{\mu_1\cdots\mu_r} d\psi^{*\mu_1}\cdots d\psi^{*\mu_r}$, we have $*\omega = \frac{\sqrt{\det(g)}}{r!(N-r)!}\omega_{\mu_1\cdots\mu_r}\epsilon^{\mu_1\cdots\mu_r}{}_{\nu_{r+1}\cdots\nu_N} d\psi^{*\nu_{r+1}}\cdots d\psi^{*\nu_N}$...

and sandwiching with $\langle n_a(r_i(t))|$ yields

$$\frac{d}{dt}\gamma_n(t) = i\langle n_a(r_i(t))|\partial_{r_i} n_a(r_i(t))\rangle \frac{dr_i}{dt} \ , \tag{1.46}$$

and so

$$\gamma_n(t) = \oint_{\mathcal{C}} i\langle n_a(r_i(t))|\partial_{r_i} n_a(r_i(t)) dr_i \ . \tag{1.47}$$

Also can have non-abelian generalization.

• **Comment:** Generally need to fix gauge (also, need multiple parameters)... Clearly, not invariant under $|n'_a(r_i(t))\rangle = e^{i\chi(r_i(t))}|n_a(r_i(t))\rangle$ since $\gamma'_n(t) = \gamma'(t) - \hbar\partial_t\chi(t)$... Take $|n_a(r_i(t))\rangle$ and $|n'_a(r_i(t))\rangle$ to be single valued so that $\chi(T) = \chi(0) + 2\pi k$ $(k \in \mathbb{Z})$... Then, $\gamma_n(t) = \gamma_n(t) - (\chi(T) - \chi(0)) = \gamma_n(t)$ is gauge invariant...

• **Comment:** Also, note that Stoke's theorem is useful here... Indeed,

$$\gamma_n(t) = \oint_{\mathcal{C}} i\langle n_a(r_i(t))|\partial_{r_i} n_a(r_i(t)) dr_i = i\int_S \left(\vec{\nabla} \times \langle n(r_i(t))|\vec{\nabla} n(r_i(t))\rangle\right) \cdot d\vec{S} \ . \tag{1.48}$$

RHS is clearly gauge invariant... However, non-trivial sometimes to evaluate RHS... So, defining $\vec{V} \equiv i\vec{\nabla} \times \langle n|\vec{\nabla} n\rangle$, we have

$$\vec{V} = i\langle\vec{\nabla} n| \times |\vec{\nabla} n\rangle = i\sum_m \langle\vec{\nabla} n|m\rangle \times \langle m|\vec{\nabla} n\rangle \tag{1.49}$$

These latter quantities can be related to matrix elements of $\vec{\nabla} H$. To see this, note that

$$\vec{\nabla}\langle m|H|n\rangle = E_n\vec{\nabla}\langle m|n\rangle = 0 = \langle\vec{\nabla} m|H|n\rangle + \langle m|\vec{\nabla} H|n\rangle + \langle m|H|\vec{\nabla} n\rangle \tag{1.50}$$

Note also that orthonormalization of the basis implies

$$\vec{\nabla}\langle m|n\rangle = \vec{\nabla}\delta_{mn} = 0 = \langle\vec{\nabla} m|n\rangle + \langle m|\vec{\nabla} n\rangle \ . \tag{1.51}$$

Plugging this into the previous equation yields

$$0 = (E_n - E_m)\langle\vec{\nabla} m|n\rangle + \langle m|\nabla H|n\rangle \ . \tag{1.52}$$

Note, by (1.51), we have that the term in (1.49) with $m = n$ vanishes since it is the cross product of a vector with itself. Therefore, we obtain

$$\vec{V} = \sum_{m\neq n} \frac{\langle n|\vec{\nabla} H|m\rangle \times \langle m|\vec{\nabla} H|n\rangle}{(E_n - E_m)^2} \ , \tag{1.53}$$

with γ the corresponding surface integral.

Exercise: Spin 1/2 particle in magnetic field, $\vec{B} \in \mathbb{R}^3$ has $H = \vec{B} \cdot \vec{\sigma}$... Here the $\vec{B}$ is the set of parameters, i.e., $B_i = r_i$ for $i = 1, 2, 3$. So, we have $\vec{\nabla} H = \vec{\sigma}$. Then, given above, check that

$$\vec{\nabla} \times \vec{A} = \frac{\vec{B}}{2B^3} \ . \tag{1.54}$$

This is the connection of a Dirac monopole... We will find a SUSY construction of this monopole soon... It will illustrate an important idea in SUSY: background fields.

• Next week we will introduce an extended version of SQM and see how Berry's phase enters. These lessons will have important implications for RG flows in higher dimensions.