1. Lecture 4: More $\mathcal{N}=2$ QM superspace and some applications

• Before continuing with our discussion of superspace and getting to some applications, let us briefly review what we learned in the last lecture.

• We started by defining multiplets of operators for the 1D particle on a line (involving $\phi(t)$ and $\psi(t)$). To make SUSY more manifest, we first introduced the Schrödinger operators

$$\psi \equiv \sqrt{\hbar}\sigma^- \ , \quad \psi^\dagger \equiv \sqrt{\hbar}\sigma^+ \ . \tag{1.1}$$

These operators satisfy the following relations

$$\{\psi,\psi\} = \left\{\psi^\dagger,\psi^\dagger\right\} = 0 \ , \quad \left\{\psi,\psi^\dagger\right\} = \hbar \ . \tag{1.2}$$

In the Heisenberg picture, these operators become field operators, $\psi(t)$. The quantities we introduced before become

$$Q = \frac{1}{\sqrt{\hbar}}\psi(W'+i\pi) \ , \quad Q^\dagger = \frac{1}{\sqrt{\hbar}}\psi^\dagger(W'-i\pi) \ , \quad H = \frac{1}{2}(\pi^2 + W'^2 - [\psi^\dagger,\psi]W'') \tag{1.3}$$

• For any field, χ, we defined the SUSY variation

$$\delta\chi = \left[\eta Q + \eta^* Q^\dagger, \chi\right] \ , \tag{1.4}$$

where η, η^* are Grassmann numbers, so they satisfy

$$\eta^2 = \eta^{*2} = 0 \ , \quad \eta\eta^* = -\eta^*\eta \ . \tag{1.5}$$

More generally, have Grassmann numbers $\eta_{1,2}$ satisfying

$$\eta_1^2 = \eta_2^2 = 0 \ , \quad \eta_1\eta_2 = -\eta_2\eta_1 \ , \quad (\eta_1\eta_2)^* = \eta_2^*\eta_1^* \ . \tag{1.6}$$

Note that we have $(\delta\chi)^\dagger = \delta\chi^\dagger$. Moreover using (1.2), (1.3), and the commutation relation $[\pi,\phi] = -i\hbar$, we saw (setting $\hbar = 1$)

$$\begin{aligned} [\eta Q,\phi] &= \eta\psi \ , \quad [\eta^* Q^\dagger,\phi] = -\eta^*\psi^\dagger \ , \quad [\eta Q,\psi] = 0 \ , \quad [\eta^* Q^\dagger,\psi] = \eta^*(W'-i\pi) \ , \\ [\eta Q,\psi^\dagger] &= \eta(W'+i\pi) \ , \quad [\eta^* Q^\dagger,\psi^\dagger] = 0 \ . \end{aligned} \tag{1.7}$$

We have

$$\delta\phi = \eta\psi - \eta^*\psi^\dagger \ , \quad \delta\psi = \eta^*(W'-i\pi) \ , \quad \delta\psi^\dagger = \eta(W'+i\pi) \ . \tag{1.8}$$

• Recall that we can obtain the Hamiltonian in (1.3) via the usual procedure of Legendre transformation[1] and canonical quantization from the following Lagrangian (**Note:** I will be a bit careless with $\psi^\dagger, \psi^*, \bar{\psi}$.)

$$S = \int dt \left[\frac{1}{2} \left(\frac{d\phi}{dt} \right)^2 + i\psi^* \frac{d\psi}{dt} - \frac{1}{2} W'^2 + \frac{1}{2} \left[\psi^*, \psi \right] W'' \right] \ . \tag{1.9}$$

• An important part of this module will be constructing SUSY-invariant Lagrangians... Superspace will be very useful here...

• We also discussed superspace: a completion of space time in which the coordinate, t, dual to H is completed by Grassmann numbers θ dual to Q and θ^* dual to $Q^\dagger$. We also constructed the most general real representation on superspace

$$\Phi(t, \theta, \theta^*) = \phi(t) + \theta\psi(t) - \theta^*\psi^*(t) + \theta\theta^* F(t) \ , \tag{1.10}$$

where there cannot be higher-order terms since $\theta^2 = (\theta^*)^2 = 0$. The fields $\phi = \phi^*$ and $F = F^*$. The fields ϕ, ψ, ψ^*, F are the "components" of the superfield, Φ.... $\phi(t)$ is often referred to as the "bottom component" of the superfield (or the "primary") and F is referred to as the "top component."

• Now, in analogy with the differential form of the Hamiltonian, let us construct the following differential operators on superspace

$$\mathcal{Q} = \frac{\partial}{\partial\theta} + i\theta^* \frac{\partial}{\partial t} \ , \qquad \mathcal{Q}^\dagger = \frac{\partial}{\partial\theta^*} + i\theta \frac{\partial}{\partial t} \ , \tag{1.11}$$

It is straightforward to verify that

$$\begin{aligned} \{\mathcal{Q}, \mathcal{Q}^\dagger\} &= \frac{\partial}{\partial\theta}\frac{\partial}{\partial\theta^*} + i\frac{\partial}{\partial t} - i\theta\frac{\partial}{\partial t}\frac{\partial}{\partial\theta} + i\theta^*\frac{\partial}{\partial t}\frac{\partial}{\partial\theta^*} - \theta^*\theta\frac{\partial^2}{\partial t^2} \\ &+ \frac{\partial}{\partial\theta^*}\frac{\partial}{\partial\theta} + i\frac{\partial}{\partial t} + i\theta\frac{\partial}{\partial t}\frac{\partial}{\partial\theta} - i\theta^*\frac{\partial}{\partial t}\frac{\partial}{\partial\theta^*} - \theta\theta^*\frac{\partial^2}{\partial t^2} \\ &= 2i\frac{\partial}{\partial t} = 2\mathcal{H} \ . \end{aligned} \tag{1.12}$$

Note also that $\{\mathcal{Q}, \mathcal{Q}\} = \{\mathcal{Q}^\dagger, \mathcal{Q}^\dagger\} = 0$ and that

$$\left(\frac{\partial}{\partial t}\right)^\dagger = -\frac{\partial}{\partial t} \ , \qquad \left(\frac{\partial}{\partial\theta}\right)^\dagger = \frac{\partial}{\partial\theta^*} \ , \tag{1.13}$$

[1] $H = \phi'\pi + \psi'\pi_\psi - \mathcal{L}$, where $\pi = \frac{\partial\mathcal{L}}{\partial\phi'} = \phi'$ and $\pi_\psi = \frac{\partial\mathcal{L}}{\partial\psi'} = -i\psi^*$.

where the first equality follows from integration by parts and the second follows from our reality conventions for Grassmann numbers.

The operators in (1.11) are useful because they allow us to define a simple action of supersymmetry on a superfield (i.e., superfields form a representation of the SUSY algebra with generators given in (1.11))

$$\delta\Phi = [\eta Q + \eta^* Q^\dagger, \Phi] = (\eta\mathcal{Q} + \eta^*\mathcal{Q}^\dagger)\Phi \ . \tag{1.14}$$

Note that

$$\begin{aligned} [\eta Q, \Phi] &= \eta(\partial_\theta + i\theta^*\partial_t)\Phi = \eta\psi - \theta^*\eta(F + i\phi') - \theta\theta^*\eta i\psi' \ , \\ [\eta^* Q^\dagger, \Phi] &= \eta^*(\partial_{\theta^*} + i\theta\partial_t)\Phi = -\eta^*\psi^* + \theta\eta^*(F - i\phi') - \theta\theta^*\eta^* i\psi'^* \ , \end{aligned} \tag{1.15}$$

Therefore, matching the variations term-by-term in the Grassmann expansion in

$$\delta\Phi = \delta\phi + \theta\delta\psi - \theta^*\delta\psi^* + \theta\theta^*\delta F \ , \tag{1.16}$$

we have, at zeroth order in the Grassmann numbers, that $\delta\phi = \eta\psi - \eta^*\psi^*$, which agrees with (1.8) (we can also check that the other SUSY variations are compatible with this discussion... you will do this more extensively on the homework).

- Also, note that matching at order $\theta\theta^*$, $\delta F = i(-\eta\psi' - \eta^*\psi'^\dagger) = -i\partial_t(\eta\psi + \eta^*\psi^\dagger)$, which is a total derivative. It is now easy to write invariant actions. To do that, let us introduce

$$\int d\theta = \int d\theta^* = \int d\theta\theta^* = \int d\theta^*\theta = 0 \ , \quad \int d\theta\theta = \int d\theta^*\theta^* = 1 \ , \quad d\theta d\theta^* = -d\theta^* d\theta \ , \tag{1.17}$$

which is equivalent to Grassmann differentiation[2]

$$\{\partial_\theta, \theta\} = \{\partial_{\theta^*}, \theta^*\} = 1 \ , \quad \{\partial_\theta, \theta^*\} = \{\partial_{\theta^*}, \theta\} = 0 \ , \tag{1.18}$$

Now, note that sums and products of real superfields are real superfields! For example,

$$\Phi^2 = \phi^2 + 2\theta\phi\psi - 2\theta^*\phi\psi^* + \theta\theta^*(2F\phi + [\psi, \psi^*]) \ , \tag{1.19}$$

and more generally

$$W(\Phi) = W(\phi) + \theta W'(\phi)\psi - \theta^* W'(\phi)\psi^* + \theta\theta^*(FW'(\phi) + \frac{1}{2}W''(\phi)[\psi, \psi^*]) \ , \tag{1.20}$$

[2]We should then define $\int d\theta\theta\theta^* = \theta^*$ and $\int d\theta^*\theta\theta^* = -\int d\theta^*\theta^*\theta = -\theta$.

Therefore, we have that

$$\delta\left(\int dt d\theta d\theta^* W(\Phi)\right) = \delta \int dt F_W = \int dt \partial_t(\cdots) = 0 \ , \tag{1.21}$$

where we have treated W as a real superfield (i.e., products and sums of real superfields are still real superfields)..

• Note that

$$\begin{aligned} \int dt d\theta d\theta^* W(\Phi) &= \int dt d\theta d\theta^* \theta\theta^* (FW'(\phi) + \frac{1}{2}W''(\phi)[\psi, \psi^*]) \\ &= -FW'(\phi) - \frac{1}{2}W''(\phi)[\psi, \psi^*] = -FW'(\phi) + \frac{1}{2}[\psi^*, \psi]W''(\phi) \ , \end{aligned} \tag{1.22}$$

which reproduces the fermionic potential terms in (1.9). The peculiar $FW'(\phi)$ term will make more sense when we add kinetic terms for ϕ and ψ, and we will see how to reproduce (1.9). These additional terms, when appropriately completed will be SUSY invariant on their own as well.

• As is often the case when we introduce new structures in physics, we need to introduce covariant derivatives to make derivatives transform in a "nice" way under the new structure. In this case, the covariant superderivatives are

$$\mathcal{D} = \partial_\theta - i\theta^*\partial_t \ , \quad \mathcal{D}^\dagger = \partial_{\theta^*} - i\theta\partial_t \ . \tag{1.23}$$

They differ from the corresponding supercharge differential operators by taking $t \to -t$. We have

$$\{\mathcal{D}, \mathcal{Q}\} = \{\mathcal{D}^\dagger, \mathcal{Q}\} = \{\mathcal{D}, \mathcal{Q}^\dagger\} = \{\mathcal{D}^\dagger, \mathcal{Q}^\dagger\} = \{\mathcal{D}, \mathcal{D}\} = \{\mathcal{D}^\dagger, \mathcal{D}^\dagger\} = 0 \ , \quad \{\mathcal{D}, \mathcal{D}^\dagger\} = -2\mathcal{H} \ . \tag{1.24}$$

and therefore

$$\mathcal{D}\Phi = \psi + \theta^*(F - i\phi') + \theta\theta^* i\psi' \ , \quad \mathcal{D}^\dagger\Phi = -\psi^\dagger - \theta(F + i\phi') + \theta\theta^* i\psi'^\dagger \ . \tag{1.25}$$

Clearly the top component is a total derivative. Therefore, $\int d\theta d\theta^* \mathcal{D}\Phi$ is not a deformation of the Lagrangian (similar statements hold for $\mathcal{D}\Phi \to \mathcal{D}^\dagger\Phi$). Note also, we have that (**Note:** the raison d'etre for covariant derivatives is that $\mathcal{D}\Phi$ should transform under SUSY in the same way as Φ)

$$\delta\mathcal{D}\Phi = [\eta Q + \eta^* Q^\dagger, \mathcal{D}\Phi] = \mathcal{D}[\eta Q + \eta^* Q^\dagger, \Phi] = \mathcal{D}(\eta\mathcal{Q} + \eta^*\mathcal{Q}^\dagger)\Phi = (\eta\mathcal{Q} + \eta^*\mathcal{Q}^\dagger)\mathcal{D}\Phi \ , \tag{1.26}$$

and so superderivatives of superfields transform like superfields under SUSY (in the second equality above, we have used the fact that Q and $Q^\dagger$ act on fields while $\mathcal{D}$ acts on coordinates; in the last equality we have used the anti-commutativity of the superderivatives and supercharges).

• Therefore, we have

$$\mathcal{S} = \int dt d\theta d\theta^* f(\Phi, \mathcal{D}\Phi, \mathcal{D}^\dagger \Phi) \ , \tag{1.27}$$

is supersymmetric for real f (this should be clear since it is a real function of superfields and hence can be written as a sum of real superfields). The most general such Lagrangian with at most two derivatives is then

$$\mathcal{S} = \int dt d\theta d\theta^* \left(-\frac{1}{2} \mathcal{D}\Phi \mathcal{D}^\dagger \Phi + W(\Phi) \right) \ . \tag{1.28}$$

It is instructive to expand this Lagrangian out in coordinates. Doing so, we obtain

$$\begin{aligned} \mathcal{S} &= \int dt \left(-\frac{1}{2}(-i\psi'(-\psi^*) + \psi(-i\psi'^*) - F^2 - \phi'^2) - W'F + \frac{1}{2} W''(\phi)[\psi^*, \psi] \right) \\ &= \int dt \left(\frac{1}{2}(\phi'^2 + F^2) + i\psi^*\psi' - W'F + \frac{1}{2} W''[\psi^*, \psi] \right) \ . \end{aligned} \tag{1.29}$$

• Note that F does not appear with a derivative: it is an auxiliary field... Its equations of motion can be solved classically (it appears quadratically)... this is called "integrating out the auxiliary field"

$$F = W'(\phi) \ , \tag{1.30}$$

which derives the identity we used before... Moreover, plugging this result into the above action gives us what we found before in (1.9). Thus, superspace gives a nice linear realization of SUSY even in interacting theories and also allows us to easily write out SUSY lagrangians.... Can go to more fields (will explore this more on homework)

$$\mathcal{S} = \int dt d\theta d\theta^* \left(-\frac{1}{2} \sum_i \mathcal{D}\Phi_i \mathcal{D}^\dagger \Phi_i + W(\Phi_i) \right) \tag{1.31}$$

• Write out the component lagrangian for (1.31) and integrate out the auxiliary fields in homework.

• The Φ multiplet we introduced above is an example of a "long" multiplet of fields (**Note: these are different multiplets than the multiplets of states we have discussed so**

far...): unless $F=0$ (e.g., if $W=0$), it has every component of superspace non-zero (this later case is an example of "superconformal quantum mechanics"... $H=\frac{d^2}{dt^2}(\phi^2)$). We can also get short multiplets. These will be useful later. An important example are chiral (anti-chiral) multiplets:

$$\mathcal{D}^\dagger X = 0 \ , \quad \mathcal{D}X^\dagger = 0 \tag{1.32}$$

Such chiral superfields are functions of $\tau = t - i\theta\theta^*$ and θ (note that $\mathcal{D}^\dagger\tau = \mathcal{D}^\dagger\theta = 0$) while such anti-chiral superfields are functions of $\tau^* = t + i\theta\theta^*$ and θ^* (note that $\mathcal{D}\tau^* = \mathcal{D}\theta^* = 0$), so

$$\begin{aligned} X &= \chi(\tau) + \theta\psi(\tau) = \chi(t) + \theta\psi(t) - i\theta\theta^*\chi'(t) \ , \\ X^* &= \chi^*(\tau) - \theta^*\psi^*(\tau) = \chi^*(t) - \theta^*\psi^*(t) + i\theta\theta^*\chi'^*(t) \ . \end{aligned} \tag{1.33}$$

• We can then construct new SUSY invariants by considering terms of the form

$$\delta\mathcal{L} = \int d\theta X + \int d\theta^* X^* \ . \tag{1.34}$$

Some such terms cannot be written as integrals over all of superspace. Indeed, it is easy to check from

$$\delta X = [\eta Q + \eta^* Q^\dagger, X] = (\eta\mathcal{Q} + \eta^*\mathcal{Q}^\dagger)X \ , \tag{1.35}$$

that $[\eta Q, \psi] = 0$ and $[\eta^* Q^\dagger, \psi] = -2i\chi'$ (which is a total derivative) and so the above is indeed an invariant (note $[\eta^* Q^\dagger, \psi^*] = 0$ and $[\eta Q, \psi^*] = 2i\chi'^*$).

• Note also from the above that

$$[Q^\dagger, \chi] = [Q, \chi^\dagger] = 0 \ . \tag{1.36}$$

These are precisely the Q-closed (anti-chiral) / $Q^\dagger$-closed (chiral) operators we encountered in our previous lecture. We learn that they are primaries (i.e., first components) of anti-chiral and chiral superfields.... We get Q-exact operators as primaries of, e.g., $\mathcal{D}\Phi$ (i.e., ψ)and $Q^\dagger$-exact operators as primaries of $\mathcal{D}^\dagger\Phi$ (i.e., ψ^*).

• Easy to show that they form a structure called a ring (known in the SUSY literature as a **"chiral ring"**)... Recall that a ring is a set equipped with addition and multiplication. Addition is associative, commutative, has a 0 element, and an inverse. Multiplication is associative and has a unit element. Finally multiplication and addition are compatible in the sense that multiplication is distributive w.r.t. addition... All of the above conditions are easily verified... It is also simple to see that

$$\mathcal{D}^\dagger X_{1,2} = 0 \Rightarrow \mathcal{D}^\dagger(X_1 + X_2) = \mathcal{D}^\dagger(X_1 X_2) = 0 \ . \tag{1.37}$$

The multiplication identity follows from the fact that $\theta^2 = 0$ (if we just expand in terms of θ and τ). These rings will play an important role in the field theories we analyze later. Needless to say, the above properties can easily be generalized to anti-chiral superfields (note that if X is chiral, then $X^\dagger$ is anti-chiral).

• *Spaces parameterized by higher dimensional analogs of these operators (e.g., moduli spaces and conformal manifolds) will naturally give rise to QM: one reason is that the corresponding operators—like QM operators—do not have singularities when we bring them together...*

• The above quantum mechanical system in (1.28) has a $U(1)_R$-symmetry, i.e., an internal $U(1)$ symmetry that doesn't commute with SUSY

$$[R, Q] = -Q \ , \quad [R, Q^\dagger] = Q^\dagger \ , \quad [R, H] = 0 \ , \tag{1.38}$$

which we define to mean that Q has R-charge -1 and $Q^\dagger$ has R-charge $+1$. It is easy to see that under this symmetry ϕ has R-charge zero, ψ has R-charge -1, and $\psi^\dagger$ has R-charge $+1$ (the auxiliary field, F has R-charge zero). Finally, note that the R-charge is

$$R = \psi^\dagger \psi \ . \tag{1.39}$$

As we will soon see, the existence of such extra symmetries in many SUSY theories will lead to powerful constraints.

• We saw the Witten index was topological: it didn't depend on explicit length scales, β (i.e., $\frac{d}{d\beta} I_W = 0$)... Now, we want to link the Witten index to topological invariants of manifolds... In order to do this, it will be helpful to formalize the relation we saw above and in the previous lecture between Q and $Q^\dagger$ and cohomology. In particular, let us first show there is a one-to-one correspondence

$$\text{SUSYgd.states} \leftrightarrow \ker(Q^\dagger)/\text{Im}(Q^\dagger) \ , \tag{1.40}$$

where $\ker(Q^\dagger)$ is made up of $Q^\dagger$-closed SUSY states (i.e., those annihilated by $Q^\dagger$). States that are $Q^\dagger$ of something else are in $\text{Im}(Q^\dagger) \subset \text{Ker}(Q^\dagger)$. The "/" means that we work modulo terms in $\text{Im}(Q^\dagger)$ (i.e., if two states in $\ker(Q^\dagger)$ differ by such terms, we identify these two states). Since $Q^{\dagger 2} = 0$, this defines some notion of cohomology.... Indeed, replace $Q^\dagger \to d$, and you will have the cohomology you were studying last semester in the Differential Geometry module. This is the same idea in a different guise.

• To see (1.40), we wish to show $Q^\dagger \chi = 0$ implies $\chi = Q^\dagger \psi$ if and only if χ is not a SUSY ground state... Suppose χ is not a SUSY groundstate. Then, the corresponding $E \neq 0$ and

$\psi = \frac{1}{2E} Q\chi \neq 0$. Acting with $Q^\dagger$ then yields $\chi = Q^\dagger \psi$. Next, let us suppose χ is a SUSY ground state. In this case, $E = 0$ but $\chi \neq Q^\dagger \psi$ since otherwise ψ would have zero energy and we would have $0 = Q^\dagger \psi = \chi$. **q.e.d.**

- Next week we will use this lemma to study the non-linear sigma model.