1. Lecture 3: $\mathcal{N}=2$ QM superspace

• Before getting to superspace, let us briefly review what we learned in the last lecture.

• First, we introduced the Witten index

$$I = \mathrm{Tr}|_{\mathcal{H}}(-1)^F e^{-\beta H} = n_B^{SUSY} - n_F^{SUSY} \ , \quad \beta > 0 \ , \tag{1.1}$$

where we argued that the index didn't depend on β or on parameters of the problem (we will come back to this statement in this lecture).

• To make things more concrete, we studied a 1D particle with supercharges given by

$$Q \equiv \sigma^- \left(W'(\phi) + i\pi\right) \ , \quad Q^\dagger \equiv \sigma^+ \left(W'(\phi) - i\pi\right) \ , \tag{1.2}$$

and a Hamiltonian

$$\left\{Q, Q^\dagger\right\} = (\pi^2 + W'^2)\mathbb{1} - \hbar W''\sigma_3 \equiv 2H \ . \tag{1.3}$$

We argued that there are classical SUSY vacua at

$$W' = 0 \ , \tag{1.4}$$

and that we could compute exact SUSY vacua by acting with the above supercharges on

$$\Omega = \begin{pmatrix} f_+(\phi) \\ f_-(\phi) \end{pmatrix} \ , \tag{1.5}$$

where $f_\pm$ is the component of the wavefunction in the $\mathcal{H}_\pm$ part of the Hilbert space.

• Acting with the supercharges yields

$$\begin{aligned} Q|\Omega\rangle &= 0 \quad \Rightarrow \quad (W'(\phi) + i\pi)f_+ = 0 \ , \\ Q^\dagger|\Omega\rangle &= 0 \quad \Rightarrow \quad (W'(\phi) - i\pi)f_- = 0 \ . \end{aligned} \tag{1.6}$$

and so

$$f_\pm = \kappa_\pm e^{\mp\frac{W}{\hbar}} \ . \tag{1.7}$$

• In class, we assumed that the spectrum of the operator ϕ was the set of real numbers, $\mathbb{R}$ (although, we will study the more general Spectrum$(\phi^i) \in \mathcal{M}$ for some more general compact Riemannian manifold, $\mathcal{M}$, soon). In this case, for polynomial W, we saw that odd polynomials lead to no exact SUSY ground state since both $f_\pm$ must blow up in one of the directions $\phi \to \infty$ or $\phi \to -\infty$. For even polynomials, we saw that there was necessarily a unique ground state: if $W = a_{2n}\phi^{2n} + \cdots$, then, if $a_{2n} > 0$, we see that $W \to \infty$ for $|\phi| \to \infty$ and so only f_+ is normalizable.... OTOH, if $a_{2n} < 0$, we see that $W \to -\infty$ as $|\phi| \to \infty$ and only f_- is normalizable... Therefore, the Witten index in these cases is just $I = \mathrm{sign}(a_{2n})$....

• As a simple example, we took the SHO with $W = \frac{\lambda}{2}(\phi - a)^2$ and $\lambda > 0$... In this case, we had a pair of shifted bosonic SHOs

$$H = \frac{1}{2}\left[\left(\pi^2 + \lambda^2(\phi - a)^2\right)\mathbb{1} - \hbar\lambda\sigma_3\right] \ . \tag{1.8}$$

• We have one classical vacuum (at $\phi = a$) and $I = +1$. Quantum mechanically, applying our general discussion, we have

$$\Omega_0 = (\kappa_+ e^{-\frac{\lambda}{2\hbar}(\phi-a)^2} \ 0)^T \ . \tag{1.9}$$

• **Make contact with what you knew before:** Each component above is related to a SHO (with a shifted vacuum) energy. Therefore, know from QM, that we have solutions

$$\Omega_{n,+} = (\omega_n \ 0)^T \ , \quad \Omega_{n,-} = (0 \ \omega_n)^T \ , \tag{1.10}$$

where $\omega_n = \langle\phi|n\rangle$ are the SHO eigenfunctions (i.e., Hermite polynomials times Gaussians... for $n = 0$, we get the above correctly). We have,

$$H\Omega_{n,\pm} = E_{n,\pm}\Omega_{n,\pm} \ , \quad E_{n,\pm} = \hbar\lambda\left(n + \frac{1}{2} \mp \frac{1}{2}\right) \ , \tag{1.11}$$

where the energy shift is due to the σ_3 shift in (1.8) proportional to fermion number. Note $E_{n,+} = \hbar\lambda n$ and $E_{n,-} = \hbar\lambda(n + 1)$. The groundstate is $E_{0,+}$ and is bosonic, while the excited states are paired up (bosonic and fermionic)—purely a matter of convention. **Note diagram in Fig. 1:**

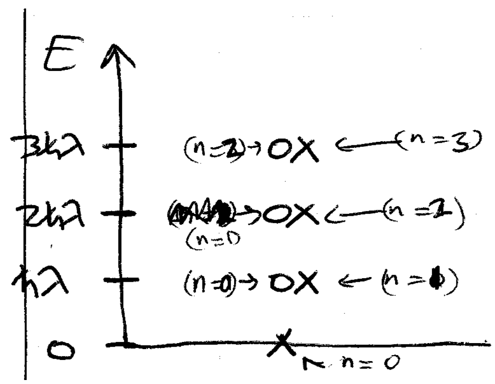

Fig. 1. From Ken Intriligator's notes: $x = +$ and $0 = -$...

In particular, we have

$$\mathrm{Tr}(-1)^F e^{-\beta H} = +1 \ , \tag{1.12}$$

Notice that the vanishing energy is an example of the cancelation we see in SUSY observables... This phenomenon is a baby version of the cancellation between fermionic and bosonic contributions to the Higgs mass alluded to in lecture 1 (again will become clearer after some conceptual / notational cleanup in later lectures).

• Note that if we then continue to $\lambda < 0$, the Witten index flips sign since the normalizable solution is now $f_- = \kappa_- e^{\frac{\lambda}{2\hbar}(\phi-a)^2}$...

• **Comment 1:** Also, good local approximation for other potentials around their local maxima and minima, i.e., around the classical vacua... We can work classically around each minimum and there

$$W'' > 0 \ . \tag{1.13}$$

The corresponding classical contribution to the Witten index is then $+1$ as in the $\lambda > 0$ case above... Around each maximum, we have

$$W'' < 0 \ . \tag{1.14}$$

The corresponding classical contribution to the Witten index is then -1 as in the $\lambda < 0$ case...

• **Comment 2:** Note that (almost!) no matter what we do for $W(\phi) = a_2\phi^2 + a_1\phi + a_0$ with, say, $a_2 > 0$ and $a_1 < 0$, we would still find a SUSY ground state above... Don't need zeros of W... $W' = 2a_2\phi + a_1$... Will intersect ϕ axis as long as $a_2 \neq 0$... But what if we set $a_2 \to 0_+$? Then, (say with $a_1 \neq 0$), the LHS of the parabola goes off to infinity and we change the potential at infinity... Change Hilbert because change boundary conditions (and make infinitely large change in energy)... Then, derivative becomes constant non-zero and no classical solutions (SUSY is broken classically... Witten index is zero)... If $a_1 = 0$, SUSY is still broken because no zero energy normalizable wavefunction...

• Now, imagine that we take $\delta W = a_3\phi^3$ with $a_3 > 0$. We end up with something **as in Fig. 2**

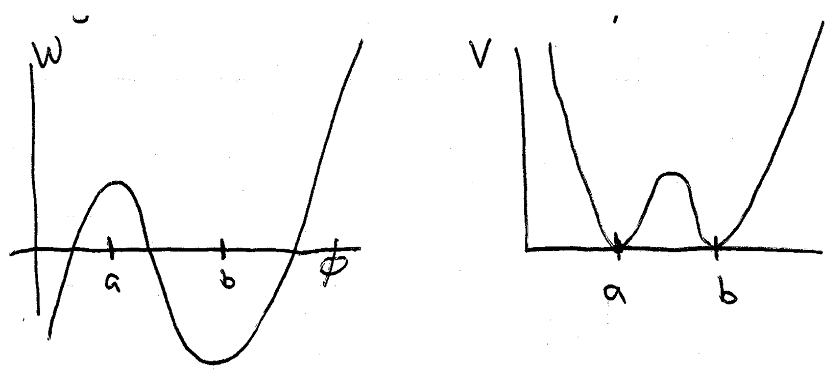

Fig. 2. From Ken Intriligator's notes

• Again, we have changed the behavior of the potential at infinity... We have brought in a classical vacuum at $\phi = a$... Think about this as two SHOs localized around a and b... In this case, a is fermionic (here $W'' < 0$, so the contribution to the Witten index is -1), while b is bosonic (here $W'' > 0$, so the contribution to the Witten index is $+1$) so the Witten index vanishes:

$$I = -1 + 1 = 0 \ , \tag{1.15}$$

where we have examined the classical solutions around $\phi = a, b$...

• So, what happens quantum mechanically? Well, we know there is no normalizable solution: the two classical short representations paired up to become a long representation and leave the zero energy part of the Hilbert space... This is done via tunneling effects that lead to non-zero energy... More precisely via objects called "instantons"... This is a fun topic I wish we could cover, but there will be lots of other fun stuff to get to.

• Note also, that we could have arranged $a \to b$ so that the two classical vacua coalesce into a single vacuum with $I = 0$ (here $W'' = 0$)... Also, we could have arranged for there to be no real solutions to $W' = 0$... This would have led to classical SUSY breaking... As opposed to above "dynamical" SUSY breaking...

• Consider now the more general

$$W = \lambda_K \phi^K + \lambda_{K-1}\phi^{K-1} + \lambda_{K-2}\phi^{K-2} + \cdots \ . \tag{1.16}$$

For $\lambda_K \neq 0$, I is independent of $\lambda_{K-1}, \lambda_{K-2}, \cdots$. However, if we tune $\lambda_K \to 0$, then I changes. For example, if K even and $\lambda_K \to 0_\pm$, then I changes from ± 1 to 0 (here I am assuming that an odd term is the leading term as we take this limit, e.g., $\lambda_{K-1} \neq 0$... For example, if an even term is the leading term in this limit then we would just get a jump directly between ± 1...).

Let's assume that $\lambda_K > 0$. Then, we have the situation in Fig. 3

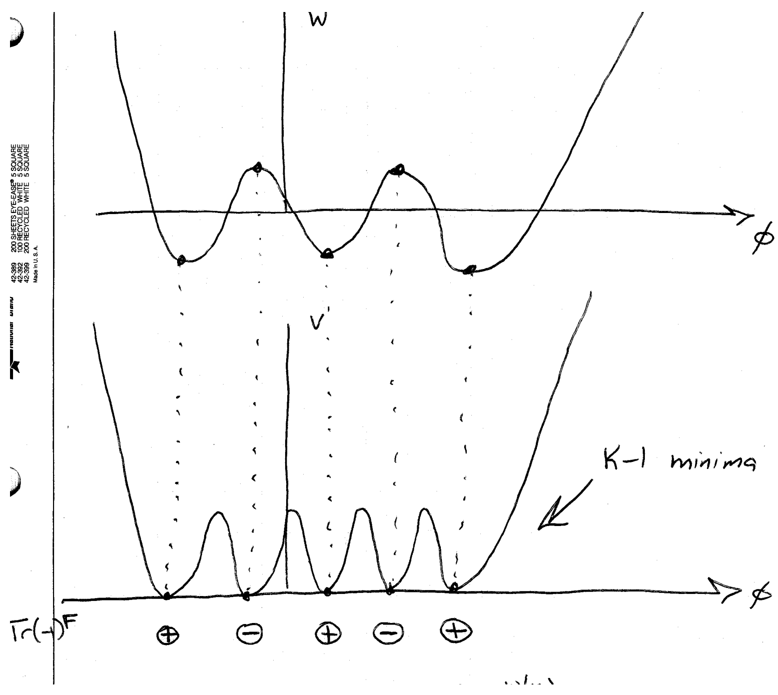

Fig. 3. From Ken Intriligator's notes

The exact groundstate is $\Omega = (e^{-\frac{W}{\hbar}} \ 0)^T$... This wavefunction is peaked at "+" vacua... Clearly, $I = +1$... This answer is exact since I doesn't depend on $\hbar$....The various $\pm$ pairs pair up (via tunneling) and become "long" multiplets of SUSY.... Again I changes as we take $\lambda_K \to 0$ with vacua moving to/from infinity.... As long as we don't do this, we can go to weak coupling and our answer doesn't change (this is a common thing that happens in SUSY).

Exercise 2.3: If $\lambda_K \to 0_\pm$ and K is odd, then what happens to the Witten index?

• At the beginning of the lecture, we argued that the Witten index is essentially independent of $\beta > 0$... Here we will get another perspective on this independence.

• To understand this statement, first recall that $H = \frac{1}{2}\left\{Q, Q^\dagger\right\} = \frac{1}{2}\left[Q, Q^\dagger\right]_+$ (here $AB \pm BA = [A, B]_\pm$) with $Q^2 = 0$. We then define

$$\Phi_\pm = [Q, \varphi_\mp]_\pm \ , \tag{1.17}$$

to be "Q-exact", where the subscript under the field corresponds to the eigenvalue of $(-1)^F$. In this sense, the Hamiltonian is Q-exact. More generally, we have operators satisfying

$$[Q, \Phi_\pm]_\mp = 0 \ , \tag{1.18}$$

that are called "Q-closed" or "(anti) chiral."

Exercise 2.2: Prove that all Q-exact operators are Q-closed. The converse is not necessarily true, i.e., there are, in general, Q-closed observables that are not Q-exact.

• **Comment:** Could have exchanged role of Q and $Q^\dagger$.

• **Aside (for those who have studied differential geometry):** This discussion should remind you of differential geometry: in particular, Q is like an exterior derivative, d (which also satisfies $d^2 = 0$). Recall that differential forms that can be written as $\omega_{k+1} = d\omega_k$ for some well-defined ω_k are called "exact" and those satisfying $d\omega = 0$ are called "closed" (they need not be exact). We have borrowed this terminology above.

• The independence of I on β now follows since

$$\begin{aligned} \frac{d}{d\beta} I &= -\mathrm{Tr}(-1)^F \exp\left(-\beta H\right) H = -\mathrm{Tr}(-1)^F \exp\left(-\beta H\right)\left(QQ^\dagger + Q^\dagger Q\right) \\ &= -\mathrm{Tr}(-1)^F \exp\left(-\beta H\right)\left(QQ^\dagger - QQ^\dagger\right) = 0 \ , \end{aligned} \tag{1.19}$$

where, in the second-to-last inequality, we have used cyclicity of the trace (which we can get by inserting complete sets of states, $\sum_a |a\rangle\langle a| = \mathbb{1}$, between each operator) and the fact that $(-1)^F Q = -Q(-1)^F$.

• We also have that other observables do not depend on β.... Treating $I = \mathrm{Tr}(-1)^F e^{-\beta H}$ as our partition function, we can also show that the one-point function of an observable, $\Phi_\pm$, is independent of T under certain conditions

$$\begin{aligned}
\frac{d}{d\beta}\langle\Phi_+\rangle &\equiv \frac{d}{d\beta}\mathrm{Tr}(-1)^F \exp(-\beta H)\Phi_+ = -\mathrm{Tr}(-1)^F \exp(-\beta H) H\Phi_+ \\
&= -\mathrm{Tr}(-1)^F \exp(-\beta H)(QQ^\dagger + Q^\dagger Q)\Phi_+ \\
&= -\mathrm{Tr}(-1)^F \exp(-\beta H)(QQ^\dagger\Phi_+ + Q^\dagger\Phi_+ Q + Q^\dagger\,[Q,\Phi_+]_-) \\
&= -\mathrm{Tr}(-1)^F \exp(-\beta H)(QQ^\dagger\Phi_+ - QQ^\dagger\Phi_+ + Q^\dagger\,[Q,\Phi_+]_-) \\
&= -\mathrm{Tr}(-1)^F \exp(-\beta H)Q^\dagger\,[Q,\Phi_+]_- \ .
\end{aligned} \tag{1.20}$$

Clearly, if the operator is Q-closed / (anti) chiral, then

$$[Q,\Phi_+]_- = 0 \ \Rightarrow \ \frac{d}{d\beta}\langle\Phi_+\rangle = 0 \ . \tag{1.21}$$

• Note that if we had taken Φ_-, then the trace would identically vanish (since the only non-zero matrix elements would be between bosonic kets and fermionic bras and vice versa... in the trace, such "off-diagonal" elements do not contribute to the trace). It is possible to generalize the above discussion to higher point functions.... We will study properties of such Q-closed /chiral operators in higher dimensions...

• Will potentially revisit above after a bit more notation and exposition... Will have a natural interpretation as a partition function on a circle of length β (with the $(-1)^F$ factor inserted to make the fermions periodic around the circle, just like the bosons)...

• The above discussion suggests that the QM we have been studying is "topological," in the sense that the partition function—and appropriately defined observables, i.e., correlation functions of the Q-closed or chiral observables— do not depend on the scale β.

• Now let's move on to superspace.... i.e., the promised SUSY completion of space-time... Nice philosophical completion: H is bosonic and generates evolution of time... Q, $Q^\dagger$ are fermionic and should generate translation in a fermionic space... Superfields can be taylor expanded in terms of component fields of bosonic and fermionic nature... Will also make it easy and systematic to construct more complicated SUSY theories...

• However, let us first revisit our two-component wavefunction example and clean up / clarify notation by making fermions more manifest... Define the Schrödinger operators

$$\psi \equiv \sqrt{\hbar}\sigma^- \ , \quad \psi^\dagger \equiv \sqrt{\hbar}\sigma^+ \ . \tag{1.22}$$

These operators satisfy the following relations

$$\{\psi, \psi\} = \{\psi^\dagger, \psi^\dagger\} = 0 \ , \quad \{\psi, \psi^\dagger\} = \hbar \ . \tag{1.23}$$

In the Heisenberg picture, these operators become field operators, $\psi(t)$. The quantities we introduced in the previous lecture become

$$Q = \frac{1}{\sqrt{\hbar}}\psi(W' + i\pi) \ , \quad Q^\dagger = \frac{1}{\sqrt{\hbar}}\psi^\dagger(W' - i\pi) \ , \quad H = \frac{1}{2}(\pi^2 + W'^2 - [\psi^\dagger, \psi]W'') \tag{1.24}$$

• For any field, χ, we define the SUSY variation

$$\delta\chi = \left[\eta Q + \eta^* Q^\dagger, \chi\right] \ , \tag{1.25}$$

where η, η^* are Grassmann numbers, so they satisfy

$$\eta^2 = \eta^{*2} = 0 \ , \quad \eta\eta^* = -\eta^*\eta \ . \tag{1.26}$$

More generally, have Grassmann numbers $\eta_{1,2}$ satisfying

$$\eta_1^2 = \eta_2^2 = 0 \ , \quad \eta_1\eta_2 = -\eta_2\eta_1 \ , \quad (\eta_1\eta_2)^* = \eta_2^*\eta_1^* \ . \tag{1.27}$$

Note that we have $(\delta\chi)^\dagger = \delta\chi^\dagger$. Moreover using (1.23), (1.24), and the commutation relation $[\pi, \phi] = -i\hbar$, we see (setting $\hbar = 1$)

$$\begin{aligned} [\eta Q, \phi] &= \eta\psi \ , \quad [\eta^* Q^\dagger, \phi] = -\eta^*\psi^\dagger \ , \quad [\eta Q, \psi] = 0 \ , \quad [\eta^* Q^\dagger, \psi] = \eta^*(W' - i\pi) \ , \\ [\eta Q, \psi^\dagger] &= \eta(W' + i\pi) \ , \quad [\eta^* Q^\dagger, \psi^\dagger] = 0 \ . \end{aligned} \tag{1.28}$$

We have

$$\delta\phi = \eta\psi - \eta^*\psi^\dagger \ , \quad \delta\psi = \eta^*(W' - i\pi) \ , \quad \delta\psi^\dagger = \eta(W' + i\pi) \ . \tag{1.29}$$

• It will be useful for us to define actions and Lagrangians for our theories—the actions will be integrals of local Lagrangians (i.e., of terms involving products of fields defined at particular space-time points) over "superspace" (a generalization of the ordinary spacetime– in this case time, t)... Also, naturally leads to relativistically invariant theories in higher dimensions... since don't pick out a preferred time direction there...

• Note that in the $\hbar \to 0$ limit, the ψ fields become just grassman numbers / functions: they are just fields taking values in the anti-commuting numbers / Grassmann numbers (all the anti-commutators in (1.23) vanish, and we write $\{\psi, \psi\} = \{\psi^*, \psi^*\} = \{\psi, \psi^*\} = 0$). We can write simple Lagrangians out of these fields.

• Note that we can obtain the Hamiltonian in (1.24) via the usual procedure of Legendre transformation[1] and canonical quantization from the following Lagrangian (**Note:** I will be a bit careless with $\psi^\dagger, \psi^*, \bar{\psi}$.)

$$S = \int dt \left[\frac{1}{2} \left(\frac{d\phi}{dt} \right)^2 + i\psi^* \frac{d\psi}{dt} - \frac{1}{2} W'^2 + \frac{1}{2} \left[\psi^*, \psi \right] W'' \right] \ . \tag{1.30}$$

Recalling $(\psi_1 \psi_2)^* = \psi_2^* \psi_1^*$, we see that S is real (recall that ϕ is real). **Note:** the ordering of fermions[2]...

• Let us consider the fluctuations around classical minima of the potential, $V = \frac{1}{2} W'^2$

$$V'|_{W'=0} = W'W'' = 0 \ , \quad V''|_{W'=0} = W''^2 \ . \tag{1.31}$$

In particular, we see that the mass of the boson is $m_\phi = W''|_{W'=0}$. This is the same as the fermionic mass if we think of the last term as a fermionic mass term, i.e., we have

$$m_\phi = W''|_{W'=0} = m_\psi \tag{1.32}$$

as discussed in lecture 1.

• It should now be clear in what sense bosonic and fermionic contributions to the SUSY SHO groundstate gave us zero energy...

• Let us now discuss superspace, i.e., the extension of regular space to include anti-commuting coordinates. Since we are doing quantum mechanics, we have a single bosonic coordinate, t (it has the usual behavior we expect for coordinates, e.g., $t^n \neq 0$ if $t \neq 0$)... This coordinate is "dual" to H (in the sense that $H = i\partial_t$ generates time evolution). So we should also have Grassmann coordinates dual to Q and $Q^\dagger$—these will be called θ and θ^*. A real superfield, Φ, is the most general real function on superspace.

$$\Phi(t, \theta, \theta^*) = \phi(t) + \theta\psi(t) - \theta^*\psi^*(t) + \theta\theta^* F(t) \ , \tag{1.33}$$

where there cannot be higher-order terms since $\theta^2 = (\theta^*)^2 = 0$. We then see that $\phi = \phi^*$ and $F = F^*$. The fields ϕ, ψ, ψ^*, F are the "components" of the superfield, Φ.... $\phi(t)$ is

[1] $H = \phi'\pi + \psi'\pi_\psi - \mathcal{L}$, where $\pi = \frac{\partial \mathcal{L}}{\partial \phi'} = \phi'$ and $\pi_\psi = \frac{\partial \mathcal{L}}{\partial \psi'} = -i\psi^*$.

[2] In the limit that the ψ, ψ^* become Grassmann functions, $[\psi^*, \psi] = 2\psi^*\psi$. Indeed, these two expressions differ by higher-order terms in $\hbar$ although they are equivalent at leading order in $\hbar$. This phenomenon is common, and such ambiguities are referred to as "contact" terms. We need to choose the form in (1.30) to preserve SUSY in the quantum theory... As we will see, such terms arise naturally in the superspace formulation.

often referred to as the "bottom component" of the superfield (or the "primary") and F is referred to as the "top component."

• Now, in analogy with the differential form of the Hamiltonian, we can construct the following differential operators on superspace

$$\mathcal{Q} = \frac{\partial}{\partial\theta} + i\theta^*\frac{\partial}{\partial t} \ , \qquad \mathcal{Q}^\dagger = \frac{\partial}{\partial\theta^*} + i\theta\frac{\partial}{\partial t} \ . \tag{1.34}$$

It is straightforward to verify that

$$\begin{aligned}\{\mathcal{Q},\mathcal{Q}^\dagger\} &= \frac{\partial}{\partial\theta}\frac{\partial}{\partial\theta^*} + i\frac{\partial}{\partial t} - i\theta\frac{\partial}{\partial t}\frac{\partial}{\partial\theta} + i\theta^*\frac{\partial}{\partial t}\frac{\partial}{\partial\theta^*} - \theta^*\theta\frac{\partial^2}{\partial t^2} \\ &+ \frac{\partial}{\partial\theta^*}\frac{\partial}{\partial\theta} + i\frac{\partial}{\partial t} + i\theta\frac{\partial}{\partial t}\frac{\partial}{\partial\theta} - i\theta^*\frac{\partial}{\partial t}\frac{\partial}{\partial\theta^*} - \theta\theta^*\frac{\partial^2}{\partial t^2} \\ &= 2i\frac{\partial}{\partial t} = 2\mathcal{H} \ .\end{aligned} \tag{1.35}$$

Note also that $\{\mathcal{Q},\mathcal{Q}\} = \{\mathcal{Q}^\dagger,\mathcal{Q}^\dagger\} = 0$ and that

$$\left(\frac{\partial}{\partial t}\right)^\dagger = -\frac{\partial}{\partial t} \ , \qquad \left(\frac{\partial}{\partial\theta}\right)^\dagger = \frac{\partial}{\partial\theta^*} \ , \tag{1.36}$$

where the first equality follows from integration by parts and the second follows from our reality conventions for Grassmann numbers.

The operators in (1.34) are useful because they allow us to define a simple action of supersymmetry on a superfield (i.e., superfields form a representation of the SUSY algebra with generators given in (1.34))

$$\delta\Phi = [\eta Q + \eta^* Q^\dagger, \Phi] = (\eta\mathcal{Q} + \eta^*\mathcal{Q}^\dagger)\Phi \ . \tag{1.37}$$

Note that

$$\begin{aligned}[][\eta Q, \Phi] &= \eta(\partial_\theta + i\theta^*\partial_t)\Phi = \eta\psi - \theta^*\eta(F + i\phi') - \theta\theta^*\eta i\psi' \ , \\ [\eta^* Q^\dagger, \Phi] &= \eta^*(\partial_{\theta^*} + i\theta\partial_t)\Phi = -\eta^*\psi^* + \theta\eta^*(F - i\phi') - \theta\theta^*\eta^* i\psi'^* \ ,\end{aligned} \tag{1.38}$$

Therefore, matching the variations term-by-term in the Grassmann expansion in

$$\delta\Phi = \delta\phi + \theta\delta\psi - \theta^*\delta\psi^* + \theta\theta^*\delta F \ , \tag{1.39}$$

we have, at zeroth order in the Grassmann numbers, that $\delta\phi = \eta\psi - \eta^*\psi^*$, which agrees with (1.29) (we can also check that the other SUSY variations are compatible with this discussion... you will do this more extensively on the homework).

• Also, note that matching at order $\theta\theta^*$, $\delta F = i(-\eta\psi' - \eta^*\psi'^\dagger) = -i\partial_t(\eta\psi + \eta^*\psi^\dagger)$, which is a total derivative. It is now easy to write invariant actions. To do that, let us introduce

$$\int d\theta = \int d\theta^* = \int d\theta\theta^* = \int d\theta^*\theta = 0 \ , \quad \int d\theta\theta = \int d\theta^*\theta^* = 1 \ , \quad d\theta d\theta^* = -d\theta^* d\theta \ , \tag{1.40}$$

which is equivalent to Grassmann integration[3]

$$\{\partial_\theta, \theta\} = \{\partial_{\theta^*}, \theta^*\} = 1 \ , \quad \{\partial_\theta, \theta^*\} = \{\partial_{\theta^*}, \theta\} = 0 \ , \tag{1.41}$$

Now, note that sums and products of real superfields are real superfields! For example,

$$\Phi^2 = \phi^2 + 2\theta\phi\psi - 2\theta^*\phi\psi^* + \theta\theta^*(2F\phi + [\psi, \psi^*]) \ , \tag{1.42}$$

and more generally

$$W(\Phi) = W(\phi) + \theta W'(\phi)\psi - \theta^* W'(\phi)\psi^* + \theta\theta^*(FW'(\phi) + \frac{1}{2}W''(\phi)[\psi, \psi^*]) \ , \tag{1.43}$$

Therefore, we have that

$$\delta\left(\int dt d\theta d\theta^* W(\Phi)\right) = \delta \int dt F_W = \int dt \partial_t(\cdots) = 0 \ , \tag{1.44}$$

where we have treated W as a real superfield (i.e., products and sums of real superfields are still real superfields)..

• Note that

$$\begin{aligned}\int dt d\theta d\theta^* W(\Phi) &= \int dt d\theta d\theta^* \theta\theta^*(FW'(\phi) + \frac{1}{2}W''(\phi)[\psi, \psi^*]) \\ &= -FW'(\phi) - \frac{1}{2}W''(\phi)[\psi, \psi^*] = -FW'(\phi) + \frac{1}{2}[\psi^*, \psi]W''(\phi) \ ,\end{aligned} \tag{1.45}$$

which reproduces the fermionic potential terms in (1.30). The peculiar $FW'(\phi)$ term will make more sense when we add kinetic terms for ϕ and ψ, and we will see how to reproduce (1.30). These additional terms, when appropriately completed will be SUSY invariant on their own as well.

• With a bit more technology under our belts next week, we will also see some interesting applications / pay off.

[3] We should then define $\int d\theta\theta\theta^* = \theta^*$ and $\int d\theta^*\theta\theta^* = -\int d\theta^*\theta^*\theta = -\theta$.