1. Lecture 2: The Witten index and some applications

Review from Lecture 1:

• Last time, we started in $0+1$ dimensions and saw that ($\mathcal{N}=2$) SUSY QM $\Rightarrow$ $H \geq 0$ since

$$\left\{Q, Q^{\dagger}\right\} \equiv QQ^{\dagger} + Q^{\dagger}Q = 2H \ , \tag{1.1}$$

and so

$$\langle\Omega|H|\Omega\rangle = \frac{1}{2}\left(\langle\Omega|QQ^{\dagger}|\Omega\rangle + \langle\Omega|Q^{\dagger}Q|\Omega\rangle\right) = \frac{1}{2}\left(|Q^{\dagger}|\Omega\rangle|^2 + |Q|\Omega\rangle|^2\right) \geq 0 \ . \tag{1.2}$$

• We saw that Q and $Q^{\dagger}$ were "fermionic" while H is bosonic.

• States with $E \neq 0$ are in "long multiplets" of SUSY since they are two dimensional (they are made up of bosonic / fermionic pairs).

• To see this, suppose we have a state with energy, $E_n \neq 0$... Can define

$$a = \frac{1}{\sqrt{2E_n}} Q \ , \quad a^{\dagger} = \frac{1}{\sqrt{2E_n}} Q^{\dagger} \ . \tag{1.3}$$

They satisfy the algebra

$$\left\{a^{\dagger}, a\right\} = 1 \ , \quad \{a, a\} = \left\{a^{\dagger}, a^{\dagger}\right\} = 0 \ . \tag{1.4}$$

This is a 2D Clifford algebra. An irreducible representation has two states, $|\pm\rangle$ with, say, $(-1)^F|\pm\rangle = \pm|\pm\rangle$

$$a|-\rangle = a^{\dagger}|+\rangle = 0 \ , \quad a|+\rangle = |-\rangle \ , \quad a^{\dagger}|-\rangle = |+\rangle \ . \tag{1.5}$$

• Moreover, zero energy states of SUSY QM are in one-to-one correspondence with states annihilated by Q and $Q^{\dagger}$. These may be bosonic OR fermionic. They form "short multiplets" of SUSY since they have dimension one...

• Recall that if there are no $E = 0$ states, then SUSY is "spontaneously broken"—the theory still has a SUSY algebra, but the ground state(s) don't preserve the supercharges. If there are $E = 0$ states, then SUSY is "preserved."

End review

• Call the number of bosonic zero energy states n_B^{SUSY} and the number of fermionic zero energy states n_F^{SUSY}. Witten defined an index (the "Witten index" [1]) that counts these states

$$I = \mathrm{Tr}(-1)^F e^{-\beta H} = n_B^{SUSY} - n_F^{SUSY} \ , \tag{1.6}$$

where the only contributions are from $H = 0$ states (and the exponential factor makes the sum well-defined). Note that (provided the theory has a discrete spectrum), the index is independent of β (we will revisit this again at the end of the lecture).

• If $I \neq 0$, SUSY is unbroken (if $I = 0$, then we can't say without more information).

• The index, I, will give us partial information about the vacuum structure of the theory (after all, it doesn't separately compute n_B^{SUSY} and n_F^{SUSY})... On the other hand, we will see that, since it is independent of most parameters (will make this precise), it is easily calculable: can make a deformation of the theory to a simpler regime and compute I...

• **Basic idea:** vacua have to pair up to form long multiplet. Long multiplets break up into pair of short multiplets as vary parameters....

• **Hidden assumption:** no new states enter the theory... i.e., the Hilbert space doesn't have new states... will see it below...

• **Example:** Let us consider a simple illustrative example: a 1D particle with two spin states. We take the wavefunction to be of the form (we replace $x \leftrightarrow \phi$ so as to make the connection to higher dimensions somewhat more manifest)

$$\Omega = (f_+(\phi) \ f_-(\phi))^T \ . \tag{1.7}$$

The first entry is the component in $\mathcal{H}_+$ and the second entry is the component in $\mathcal{H}_-$, where the Hilbert space factorizes as $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $\mathcal{H}_\pm$ refers to the part of the Hilbert space with states of Fermion number $(-1)^F = \pm 1$.... Recall that the conjugate momentum is $\pi = -i\hbar\partial_\phi$ (i.e., $[\pi, \phi] = -i\hbar$). Let us define

$$Q \equiv \sigma^- \left(W'(\phi) + i\pi\right) \ , \quad Q^\dagger \equiv \sigma^+ \left(W'(\phi) - i\pi\right) \ , \tag{1.8}$$

where

$$\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ . \tag{1.9}$$

As a result, we have

$$\left\{Q, Q^\dagger\right\} = (\pi^2 + W'^2)\mathbb{1} - \hbar W''\sigma_3 \equiv 2H \ , \tag{1.10}$$

where (up to an overall sign that amounts to **a choice of convention**)

$$(-1)^F = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{1.11}$$

Since the supercharges are acting as raising and lowering operators, it is also natural to call σ_3 fermion number.

• Note that $Q^2 = (Q^\dagger)^2 = 0$ since $(\sigma^\pm)^2 = 0$. Note also that the first two terms in (1.10) are the kinetic and potential energies, and the last term has the form of the interaction of a spin with a magnetic field of strength W'' (note also the relation between the potential and the magnetic field required by SUSY—they are tied together by the superpotential, W).

• The first two contributions in (1.10) are, in some sense, bosonic (they only involve ϕ and derivatives w.r.t. ϕ), whereas the last contribution, proportional to fermion number, is due to fermions (will become more manifest soon).

Exercise 2.1: Check that $[Q, H] = \left[Q^\dagger, H\right] = 0 = [(-1)^F, H]$. Also, check that $(-1)^F Q = -Q(-1)^F$ and $(-1)^F Q^\dagger = -Q^\dagger(-1)^F$

In order for the spectrum of H to be discrete for non-compact field space $\mathbb{R} \ni \phi$, we assume that

$$\lim_{|\phi|\to\infty} |W'(\phi)| = \infty \tag{1.12}$$

In the weak coupling / classical limit, $\hbar \to 0$, we ignore all the terms in (1.10) except W'^2, and we try to minimize this term. So, at "tree level" (another way of saying classically), the number of SUSY groundstates is just the number of zeros of

$$W'(\phi) = 0 \ . \tag{1.13}$$

• *The solutions to* (1.13) *are the classical SUSY vacua.* Also, the values of ϕ for which (1.13) holds are the locations of the classical vacua: they are the places where the particle wants to live in the $\hbar \to 0$ limit...

• Since the index doesn't change under deformation of $\hbar$, we can trust that the Witten index computed for small $\hbar$ gives the correct quantum Witten index... This is an example of the power of SUSY... If the Witten index in this limit is non-zero, then SUSY is preserved even quantum mechanically... If it is zero, all bets are off...

Question: Do we have an exact supersymmetric groundstate? We have that

$$\begin{aligned} Q|\Omega\rangle &= 0 \quad \Rightarrow \quad (W'(\phi) + i\pi)f_+ = 0 \ , \\ Q^\dagger|\Omega\rangle &= 0 \quad \Rightarrow \quad (W'(\phi) - i\pi)f_- = 0 \ . \end{aligned} \tag{1.14}$$

Note that these are first-order differential equations as opposed to second order (indication of the power of SUSY)... Solutions

$$f_\pm = \kappa_\pm e^{\mp \frac{W}{\hbar}} \ . \tag{1.15}$$

• In light of (1.12), to make f_+ normalizable, we should **(1)** choose $\lim_{\phi\to\pm\infty} W(\phi) = +\infty$. However, this will make f_- blow up. On the other hand, to make f_- normalizable, we should **(2)** take $\lim_{\phi\to\pm\infty} W(\phi) = -\infty$, but this will make f_+ blow up. Finally, if we have $\lim_{\phi\to\infty} W(\phi) = -\lim_{\phi\to-\infty} W(\phi)$, then neither f_+ nor f_- are normalizable. In case **(3)** there is no SUSY ground state (i.e., SUSY is broken). On the other hand, in cases **(1), (2)** we have a unique SUSY vacuum.

• In the $\hbar \to 0$ limit, f_+ peaks strongly around local minima of W (i.e., solutions to (1.13) with $W'' > 0$) while f_- peaks strongly around local maxima of W (i.e., solutions to (1.13) with $W'' < 0$).

• Let us illustrate these ideas with a simple example: the SUSY simple harmonic oscillator (SHO)

$$W(\phi) = \frac{\lambda}{2}(\phi - a)^2 \ , \quad \lambda > 0 \ . \tag{1.16}$$

Note: Unique classical vacuum at $\phi = a$ (since $W' = \lambda(\phi - a)$). Witten index therefore must be non-zero and actually have absolute value 1 (in our conventions will be $I = +1$, see below)... So expect an exact vacuum....

• Indeed, we have

$$H = \frac{1}{2}\left[\left(\pi^2 + \lambda^2(\phi - a)^2\right)\mathbb{1} - \hbar\lambda\sigma_3\right] \ . \tag{1.17}$$

Applying our general discussion, we have

$$\Omega_0 = (\kappa_+ e^{-\frac{\lambda}{2\hbar}(\phi - a)^2} \ 0)^T \ . \tag{1.18}$$

Since we are assigning $(-1)^F = +1$ to the first entry of the vector (i.e., the state is in the $\mathcal{H}_+$ part of the Hilbert space) we see that the Witten index $I = +1$ (there is one vacuum and it is bosonic).

• **Make contact with what you knew before:** Each component above is related to a SHO (with a shifted vacuum) energy. Therefore, know from QM, that we have solutions

$$\Omega_{n,+} = (\omega_n \ 0)^T \ , \quad \Omega_{n,-} = (0 \ \omega_n)^T \ , \tag{1.19}$$

where $\omega_n = \langle \phi | n \rangle$ are the SHO eigenfunctions (i.e., Hermite polynomials times Gaussians... for $n = 0$, we get the above correctly). We have,

$$H\Omega_{n,\pm} = E_{n,\pm}\Omega_{n,\pm} \ , \quad E_{n,\pm} = \hbar\lambda \left(n + \frac{1}{2} \mp \frac{1}{2} \right) \ , \tag{1.20}$$

where the energy shift is due to the σ_3 shift in (1.17) proportional to fermion number. Note $E_{n,+} = \hbar\lambda n$ and $E_{n,-} = \hbar\lambda(n+1)$. The groundstate is $E_{0,+}$ and is bosonic, while the excited states are paired up (bosonic and fermionic)—purely a matter of convention. **Note diagram in Fig.2:**

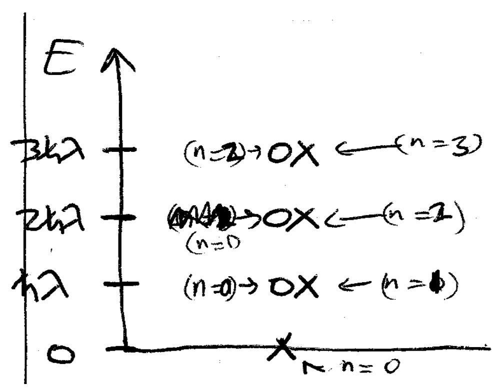

Fig. 2. From Ken Intriligator's notes: $x = +$ and $0 = -$...

In particular, we have

$$\mathrm{Tr}(-1)^F e^{-\beta H} = +1 \ , \tag{1.21}$$

Notice that the vanishing energy is an example of the cancelation we see in SUSY observables... This phenomenon is a baby version of the cancellation between fermionic and bosonic contributions to the Higgs mass alluded to in lecture 1 (again will become clearer after some conceptual / notational cleanup in later lectures).

• Note that if we then continue to $\lambda < 0$, the Witten index flips sign since the normalizable solution is now $f_- = \kappa_- e^{\frac{\lambda}{2\hbar}(\phi - a)^2}$...