1. Lecture 11

• Let us briefly review what we saw in the last lecture. We studied the interacting WZ model gotten by deforming the free chiral superfield by the relevant perturbations

$$\delta W = m\Phi^2 + \lambda\Phi^3 \ . \tag{1.1}$$

When $\lambda \neq 0$, we have interactions (this term is also relevant since $R_0(\lambda) = \frac{1}{2} = \Delta(\lambda)$). Using selection rules under (broken) symmetries we constrained the renormalization of this theory.

• For both $m, \lambda \neq 0$, there is no $U(1)_R$ symmetry and no $U(1)$ flavor symmetry. However, we can promote these broken symmetries to spurious symmetries by allowing the couplings to transform. Doing so, we have $\mathcal{J}(\phi) = \mathcal{J}(\psi_\alpha) = \mathcal{J}(F) = +1$ with $\mathcal{J}(m) = -2$ and $\mathcal{J}(\lambda) = -3$. We also have $R(\phi) = \frac{2}{3}$, $R(\psi) = -\frac{1}{3}$, and $R(F) = -\frac{4}{3}$ with $R(m) = \frac{2}{3}$ and $R(\lambda) = 0$.

• Therefore, the quantum superpotential must have the form

$$W = m\Phi^2 \cdot f\left(\frac{\lambda\Phi}{m}\right) \ , \tag{1.2}$$

where $\mathcal{J}(\lambda\phi/m) = \mathcal{J}(\lambda) + \mathcal{J}(\phi) - \mathcal{J}(m) = -3 + 1 + 2 = 0$ and $R(\lambda\phi/m) = R(\lambda) + R(\phi) - R(m) = 0 + \frac{2}{3} - \frac{2}{3} = 0$ while $\mathcal{J}(m\Phi^2) = 2$ and $R(m\Phi^2) = 2$.

• Denote $u = \lambda\Phi/m$. By studying the $u \to 0$ and $u \to \infty$ limits and by using the symmetries at $|u| = \infty$, we argued that quantum corrections / renormalization do not change W, so we have

$$W = m\Phi^2 + \lambda\Phi^3 \ , \tag{1.3}$$

in the full quantum theory as well.

• Note, however, that the Kähler potential is in general renormalized, although the Kähler metric remains positive semi definite, $g_{\phi\bar{\phi}} = \partial_\phi \bar{\partial}_{\bar{\phi}} K > 0$ since kinetic terms still need to have the correct sign.

• Let us study this theory in more detail. To that end, we can find the SUSY vacua by solving the F-term EOM and setting the result to zero

$$\bar{F} = -g^{X\bar{X}}(2m\phi + 3\lambda\phi^2) = 0 \ . \tag{1.4}$$

Since $g^{X\bar{X}} > 0$, we have that

$$2m\phi + 3\lambda\phi^2 = 0 \ . \tag{1.5}$$

The solutions describe the vacua of the theory

$$\langle\phi\rangle = -\frac{2m}{3\lambda} \ , \quad \langle\phi\rangle = 0 \ . \tag{1.6}$$

If both $m, \lambda \neq 0$, these are two separate vacua. The theory is trivial in each of these vacua (the chiral multiplet has a mass m in both vacua). If we set $\lambda = 0$, the first vacuum goes off to infinity and we have just the vacuum at $\langle\phi\rangle = 0$. If we set $m = 0$, we have a (double) vacuum at $\langle\phi\rangle = 0$... This is an interacting SCFT with $R_0^{IR}(\phi) = 2/3 = \Delta(\phi)$... In both cases, the moduli space (space of vacua) is just a point... We can integrate out the field by setting it to one of the values in (1.6).

• As we saw last time, we can also apply identical non-renormalization arguments to theories with non-trivial moduli spaces... For example, consider the following

$$\mathcal{L} = -\int d^4\theta(\bar{X}X + \bar{Y}Y + \bar{Z}Z) + \left(\int d^2\theta\lambda XYZ + \text{h.c.}\right) \ . \tag{1.7}$$

We have the following set of vacua

$$\bar{F}_X = -g^{X\bar{X}}\lambda YZ = 0 \ , \quad \bar{F}_Y = -g^{Y\bar{Y}}\lambda XZ = 0 \ , \quad \bar{F}_Z = -g^{Z\bar{Z}}\lambda XY = 0 \ . \tag{1.8}$$

Note that $g^{X\bar{X}} = g^{Y\bar{Y}} = g^{Z\bar{Z}} > 0$ is a non-trivial function of $\lambda\bar{\lambda}$ (off diagonal components of the metric vanish by the $U(1)^2 \times S_3$ flavor symmetry of the theory... We have the following three solutions

$$\begin{aligned} \langle X\rangle &= \langle Y\rangle = 0 \ , \quad \langle Z\rangle \in \mathbb{C} \ , \\ \langle X\rangle &= \langle Z\rangle = 0 \ , \quad \langle Y\rangle \in \mathbb{C} \ , \\ \langle Y\rangle &= \langle Z\rangle = 0 \ , \quad \langle X\rangle \in \mathbb{C} \ . \end{aligned} \tag{1.9}$$

These are three "branches..." Each parameterized by vevs for X, Y, and Z... They meet at the point where $\langle X\rangle = \langle Y\rangle = \langle Z\rangle = 0$ is an interacting SCFT. This is called the "origin of the moduli space." On each of these branches, we have massless modes... As we will see, these are related to spontaneously broken symmetries (i.e., symmetries preserved by the theory but broken by the vacuum)... These are called Goldstone bosons (and their super-partners)... **For example, on each branch, we break a $U(1)_R$ under which $R(X) = R(Y) = R(Z) = \frac{2}{3}$, so we have a multiplet corresponding to this broken generator (note that this is the superconformal $U(1)_R$ since it is invariant under the S_3 that permutes the fields)...**

• But we can make this more precise. There is a theorem governing such things: it is called Goldstone's theorem.... How do we see this? Suppose we have an operator, $\mathcal{O}$, charged under some global symmetry (it may be flavor or R-symmetry.... note that this reasoning is more general than SUSY) such that $\langle\mathcal{O}\rangle = \langle 0|\mathcal{O}|0\rangle \neq 0$. In this case

$$\langle 0|[Q,\mathcal{O}]|0\rangle = \mathcal{C} \neq 0 \ . \tag{1.10}$$

We can re-write this as (using a sum over a complete set of states)

$$\begin{aligned}
\mathcal{C} &= \sum_n \int d^2x \Big(\langle 0|j_0(x)|n\rangle\langle n|\mathcal{O}|0\rangle - \langle 0|\mathcal{O}|n\rangle\langle n|j_0(x)|0\rangle \Big) \\
&= \sum_n (2\pi)^2\delta^2(\mathbf{p}) \Big(\langle 0|j_0(0)|n\rangle\langle n|\mathcal{O}|0\rangle e^{-i\omega_n t} - \langle 0|\mathcal{O}|n\rangle\langle n|j_0(0)|0\rangle e^{i\omega_n t} \Big) \ .
\end{aligned} \tag{1.11}$$

Currents are still conserved even when the symmetry is spontaneously broken since this is just a choice of vacuum and $\partial_\mu j^\mu = 0$ is an operator equation. Therefore

$$\partial^0\mathcal{C} = \sum_n (2\pi)^2\delta^2(\mathbf{p}) \Big(-i\omega_n\langle 0|j_0(0)|n\rangle\langle n|\mathcal{O}|0\rangle e^{-i\omega_n t} - i\omega_n\langle 0|\mathcal{O}|n\rangle\langle n|j_0(0)|0\rangle e^{i\omega_n t} \Big) \tag{1.12}$$

Therefore we see that the states that have non-zero matrix elements (these must exist) have $\omega_n = 0$. This is what we expect for a free massless particle ($\omega_n = \mathbf{p} = 0$).

• Now, let us suppose we move onto the branch with $\langle X\rangle \neq 0$ (and all other vevs vanishing)... Since X is charged under, e.g., a $U(1)_R$ symmetry with $R(X) = 2/3$, we expect a goldstone boson in the IR. Actually, we expect a Goldstone boson in a supermultiplet... These multiplets will be free in the IR... Let us set $\langle X\rangle = x$. Then, we have $X = \langle X\rangle + \delta X \to \langle X\rangle + X$ (where, by abuse of notation, we have renamed the fluctuation about the vacuum X as well)

$$W = \lambda xYZ + \lambda XYZ \ . \tag{1.13}$$

The result is that the Y and Z multiplets acquire a mass λx. At energies below this scale we are left with a theory of the X superfield. We can integrate the massive Y and Z fields

$$\bar{F}_Y = -g^{Y\bar{Y}}(\lambda xZ + \lambda XZ) = 0 \ , \quad \bar{F}_Z = -g^{Z\bar{Z}}(\lambda xY + \lambda XY) = 0 \ . \tag{1.14}$$

From these equations, we conclude

$$\lambda xZ + \lambda XZ = \lambda xY + \lambda XY = 0 \ . \tag{1.15}$$

Since the Y and Z fields are massive, we integrate them out and plug the above EOM back into the superpotential yields

$$W = 0 \ . \tag{1.16}$$

In other words, in the deep IR, we see that the theory consists of a single free chiral multiplet, X. This is the goldstone multiplet in question. Note that this branch of the moduli space has complex dimension 1 since $x \in \mathbb{C}$ is unconstrained.

• We have a similar situation on the other branches (i.e., on the $\langle Y\rangle \neq 0$ branch, Y is the Goldstone multiplet while X and Z are massive and get integrated out... on the $\langle Z\rangle \neq 0$ branch, Z is the Goldstone multiplet while X and Y are massive and get integrated out...)... At the origin we have the interacting SCFT with $\Delta(X) = \Delta(Y) = \Delta(Z) = \frac{2}{3}$... See the attached figure...

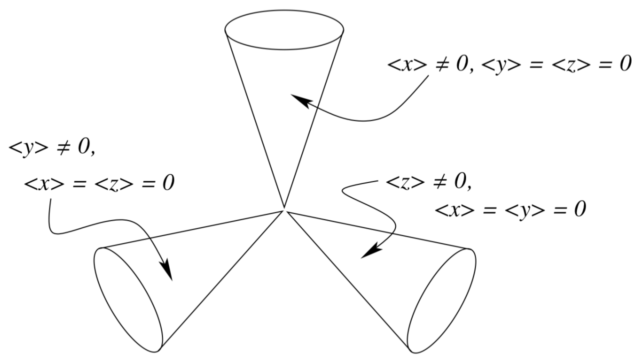

Fig. 1: The moduli space of the XYZ model.

• Now that we have a better picture of the IR behavior of the XYZ model, let us turn to the IR behavior of SQED.

• Recall that SQED has the form

$$\begin{aligned}
\mathcal{L}_{\rm SQED} &= -\int d^4\theta \sum_{i=1}^{N_f} (\bar{q}^i e^{2V} q_i + \bar{\tilde{q}}^i e^{-2V}\tilde{q}_i) - \frac{1}{g^2}\int d^2\theta d^2\bar{\theta}\Sigma^2 = \frac{1}{g^2}\Big(\frac{1}{2}D^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\
&- \partial^\mu\sigma\partial_\mu\sigma + i\lambda\gamma^\mu\partial_\mu\bar{\lambda}\Big) + \sum_i (|F_i|^2 - D_\mu\bar{\rho}^i D^\mu\rho_i + i\psi_i\gamma^\mu D_\mu\bar{\psi}^i + |\tilde{F}_i|^2 - D_\mu\bar{\tilde{\rho}}^i D^\mu\tilde{\rho}_i \\
&+ i\tilde{\psi}_i\gamma^\mu D_\mu\bar{\tilde{\psi}}^i - \sigma^2(|\rho_i|^2 + |\tilde{\rho}_i|^2) - D(|\rho_i|^2 - |\tilde{\rho}_i|^2) - i\sigma(\psi_i\bar{\psi}^i - \tilde{\psi}_i\bar{\tilde{\psi}}^i \\
&- \sqrt{2}i(\lambda\psi_i\bar{\rho}^i - \lambda\tilde{\psi}_i\bar{\tilde{\rho}}^i) - \sqrt{2}i(\bar{\lambda}\bar{\psi}^i\rho_i - \bar{\lambda}\bar{\tilde{\psi}}^i\tilde{\rho}_i))\ . \qquad (1.17)
\end{aligned}$$

In what follows, we will specialize to the case $N_f = 1$... Recall that this theory has a $U(1) \times U(1)$ flavor symmetry (the first factor is a topological symmetry under which the scalar dual to the photon shifts... the second factor is the $U(1)$ flavor symmetry that rotates the q and $\tilde{q}$ mutliplets all by the same phase).

• Note that there are zero energy solutions gotten by setting the matter fields to have vanishing vevs, $\langle\rho_i\rangle = \langle\tilde{\rho}_i\rangle = 0$. We can also set $\langle\sigma\rangle \neq 0$. This clearly gives masses to all the matter fields while leading to

$$\frac{1}{g^2}D - |\rho|^2 + |\tilde{\rho}|^2 = 0 \ , \quad D = 0 \ , \quad F = \tilde{F} = 0 \ . \tag{1.18}$$

This is what we want: descendant fields like D and F cannot acquire vevs in a SUSY vacuum (since they are Q or $\bar{Q}$ of something, this would imply that the vacuum is not annihilated by the SUSY charges, i.e., SUSY is spontaneously broken). This procedure leaves over the vector multiplet as a massless degree of freedom... We can think of this multiplet as a Goldstone multiplet for the topological symmetry... Recall that there is a dual scalar to the photon, φ, that is in a multiplet with σ... This is called the "*Coulomb* branch" of the theory... The reason is that there is a free $U(1)$ gauge field left over in the IR...

• Note that we also may have branches where we give the matter fields vevs... Indeed, we need only choose

$$\langle\sigma\rangle = 0 \ , \quad |\langle\rho\rangle|^2 = |\langle\tilde{\rho}\rangle|^2 \ . \tag{1.19}$$

• How should we understand this branch? Via the Higgs mechanism... Recall that a photon has two degrees of freedom in 4D (naively it has four, but recall that A_0 does not have a kinetic term and so is determined in terms of the other d.o.f.'s... also, we can absorb another d.o.f. via gauge transformations) and 1 degree of freedom in 3D (we lose degrees of freedom in the same way... and we start naively from 3 d.o.f.'s)...

• Consider the following non-SUSY Lagrangian with a complex scalar, ϕ, charged under a $U(1)$ gauge symmetry

$$\mathcal{L} = -D_\mu\bar{\phi}D^\mu\phi = -(\partial_\mu - iqA_\mu)\bar{\phi}(\partial^\mu + iqA^\mu)\phi \ . \tag{1.20}$$

Under a $U(1)$ gauge transformation, $\phi \to e^{-iq\Lambda}\phi$ and $A_\mu \to A_\mu + \partial_\mu\Lambda$... Gauge symmetry prevents the existence of a mass for the photon since $\frac{m^2}{2}A^\mu A_\mu$ is not gauge invariant... When we take $\langle\phi\rangle = v \neq 0$, however we do get a mass... Take $\phi = (v+\rho)e^{i\theta}$. We then have

$$\mathcal{L} \supset q^2|v|^2A^\mu A_\mu \ . \tag{1.21}$$

The reason is that ϕ transforms in such a way that it maintains gauge invariance...

• What of the SUSY example? Well, we can try to turn on a vev $\langle \rho \rangle = v \neq 0$. The D-term equations force us then to set $\langle \tilde{\rho} \rangle = ve^{i\theta}$ as well (for simplicity, let's set $\theta = 0$)... This is called the "*Higgs* branch"...

• What do we expect on general grounds? We know there is a $U(1)$ flavor symmetry that rotates all the squark / matter superfields by the same phase... So Goldstone's theorem guarantees a multiplet of Goldstone bosons... Also, we know that if the vector field gets a mass, then so too must the σ field and the gauginos, $\bar{\lambda}$. This multiplet has the same number of d.o.f.'s as a chiral multiplet... So, we need to marry it with another chiral multiplet... This should leave over a chiral multiplet comprising the $U(1)$ Goldstone multiplet...

• To see this more explicitly, note that substituting the vevs into the above Lagrangian yields the following mass terms

$$\mathcal{L}_{SQED} \supset \sqrt{2} i \bar{v} \lambda (\psi - \tilde{\psi}) \tag{1.22}$$

It is easy to see this is the only mass term involving λ... So, the gauge multiplet must get massive by eating the $q - \tilde{q}$ chiral multiplet... Note that the other linear combination $\psi + \tilde{\psi}$ is massless... This is part of the $q + \tilde{q}$ Goldstone multiplet... Note that it is natural for this multiplet to contain opposite gauge charges: this symmetry is broken. On the other hand, note that both q and $\tilde{q}$ have the same charge under the global $U(1)$ symmetry...

• Note here that there are no mixed branches: we cannot simultaneously turn on vevs for σ and the matter field primaries! That is to say: we have a 1 complex dimensional moduli space with different branches: the Coulomb and Higgs branches... The absence of mixed branches is related to what we saw in the $\mathcal{N} = (2,2)$ SQM dimensional reduction of this theory: we couldn't turn on real masses and holomorphic masses at the same time... Thinking of these objects as dynamical fields would have corresponded to going onto Coulomb and Higgs branches at the same time...

• Let us analyze the Coulomb branch more carefully... As we discussed in lecture 8, the scalar φ dual to the 3D photon is compact... It lives on a circle of length $2\pi g^2$, where g is the gauge coupling... Now, as long as there is charged matter, this coupling—as in 4D— runs... However, we have seen that a vev $\langle \sigma \rangle = v \neq 0$ gives mass to the matter fields. Below this scale the coupling therefore does not run... In particular, we have

$$\frac{1}{g_L^2} = \frac{1}{g_0^2} + \frac{1}{\langle |\sigma| \rangle} \ . \tag{1.23}$$

For large $\langle\sigma\rangle$, we have that $g_L \sim g_0$... For small vev, however, we have $g \to 0$... In particular, the Coulomb branch then shrinks to zero size... It opens up again when we switch signs of $\langle\sigma\rangle$... Therefore, we really have two Coulomb branches and a higgs branch..

• This has exactly the structure of the XYZ moduli space... Also, the symmetries match: they are both $U(1) \times U(1)$... That's because these theories are actually dual–ie.., the same in the IR!!!

• There are many other checks of this idea... For example, although our methods don't let us select the correct superconformal R symmetry for the IR SQED theory (we can only say that, based on charge conjugation invariance, $R_0^{IR}(q) = R_0^{IR}(\tilde{q})$...)... More modern techniques do, however, allow us to do this. We can check that $R_0^{IR}(q) = R_0^{IR}(\tilde{q}) = 1/3$, so we can get matches of gauge invariant operators as well. For example, $M = q\tilde{q}$ can be mapped onto one of X, Y, Z (there are other operators that get mapped onto the remaining fields).... This is the simplest example of duality!!!

• While we have reached the end of the course, it's clear there is much, much more to explore... I hope to have given you just a taste of the beauty behind SUSY and formal aspects of QFT.