1. Lecture 10

• Let us briefly review what we saw in the last lecture. We studied the structure of theories with conformal symmetry and SUSY, and saw that we necessarily got superconformal symmetry (so called superconformal field theories or SCFTs). One important aspect of these theories was that they had a superconformal R symmetry, R_0. As we will see, this is a genuine symmetry as opposed to an automorphism.

• Under this symmetry, chiral operators satisfy the following

$$R_0(\mathcal{O}) = \Delta(\mathcal{O}) \ , \quad [\bar{Q}_{\dot\alpha}, \mathcal{O}] = 0 \ . \tag{1.1}$$

In the free chiral multiplet example, $R_0(\phi) = \frac{1}{2}$... We can define composite chiral operators without the usual subtraction of singularities, i.e., $\lim_{x\to y} \mathcal{O}_1(x)\mathcal{O}_2(y) = \mathcal{O}_1\mathcal{O}_2(y)$, and dimensions of operators just add.

• **Comment:** Note that the superconformal R symmetry is a genuine symmetry of the theory as opposed to being just an automorphism of the algebra. For example, we could consider turning on

$$W = m\Phi^2 + \lambda\Phi^3 \ . \tag{1.2}$$

This theory doesn't have an R-symmetry (although the SUSY algebra has a $U(1)_R$ automorphism)... It can't because the integration measure $d^2\theta$ has $R = -2$, and we can't simultaneously have $R(\phi) = 1$ and $R(\phi) = 2/3$. It also doesn't have a superconformal R-symmetry...

• The above theory therefore cannot be both supersymmetric and conformal (the corresponding algebra does not close if we include both SUSY and conformal generators). Indeed, it is not conformal since W has scaling dimension two and so m has scaling dimension 1 and λ has scaling dimension 1/2. We will see soon using some fancy arguments that it is necessarily SUSY (i.e., has a SUSY ground state even in the quantum theory).

• Let us now understand non-conformal theories better. The simplest thing to do is to start with a free chiral multiplet and turn on

$$\delta W = m\Phi^2 \ . \tag{1.3}$$

This is a mass term. It also breaks the superconformal R symmetry since $R_0(\Phi^2) = 1 \neq 2$ (i.e., $R_0(m) = 1$). There is, however, a different R-symmetry, $R_{\frac{1}{2}}$ (where $R_{\frac{1}{2}}(\Phi) = 1$).

• What does the propagator look like in this theory? Say we are in Euclidean space. Then, we have (up to an overall factor)

$$\langle\phi(k)\phi(-k)\rangle = \frac{1}{k^2+m^2} \tag{1.4}$$

Clearly, in the limit $k \gg m$, we have that

$$\langle\phi(k)\phi(-k)\rangle \sim \frac{1}{k^2}\ . \tag{1.5}$$

This limit is called the UV (ultraviolet) or high energy / momentum limit. In this limit, the theory looks like a free massless scalar. On the other hand, for $k \ll m$, we are in the IR (infrared) or low energy / momentum limit. In this limit, we have that

$$\langle\phi(k)\phi(-k)\rangle \sim \frac{1}{m^2}\ , \tag{1.6}$$

and there is no energy to excite modes of the field...

• It is useful to Fourier transform the above to position space. In particular, we have $\langle\phi(x)\phi(y)\rangle$, and, for $|x-y| \ll m$ (the UV or short-distance limit), we have

$$\langle\phi(x)\phi(y)\rangle = \frac{1}{|x-y|}\ , \quad |x-y| \ll m^{-1}\ . \tag{1.7}$$

On the other hand, in the IR (long-distance limit), we have

$$\langle\phi(x)\phi(y)\rangle = \delta^3(x-y)\ , \quad |x-y| \gg m^{-1}\ . \tag{1.8}$$

The first limit is the CFT limit of the free scalar, while the second limit is clearly trivial (there is no propagation of fields...)... The theory is completely massive... In particular, in this regime, m^{-1} provides a short-distance cut-off, so we never should consider the divergence when $x \to y$ (this divergence is an example of something called a local or "contact" term—it is related to the UV definition of the theory) **Exercise:** Perform the Fourier transform of the momentum-space 2-pt function and check that it interpolates between these two limits.

• Therefore, our theory interpolates between an SCFT in the UV and a trivial theory in the IR.

• This is common in QFT: in the UV we start with something that has more degrees of freedom, and in the IR we end up with something with fewer degrees of freedom. In 2, 3, and 4 dimensions there are well-known theorems that formalize this idea [1].

• To codify this discussion, we introduce an energy momentum scale, μ—this is the scale at which we "observe" the theory... It is called the "RG scale"... When $\mu \to 0$ we go to the trivial theory and when $\mu \to \infty$ we get the UV CFT... To see the relative importance of couplings in the IR and UV, we define a dimensionless coupling

$$\hat{m} = \frac{m}{\mu} \ . \tag{1.9}$$

The importance of this coupling with scale is measured by the beta function

$$\beta_m \equiv \mu \frac{\partial \hat{m}}{\partial \mu} = -\hat{m} \ . \tag{1.10}$$

This means that as $\mu \to 0$, $\hat{m} \to \infty$ while for $\mu \to \infty$, $\hat{m} \to 0$... This is what we expect: in the UV limit, the masses are not important... The opposite is true in the IR.

• At scales $\mu \ll m$, we cannot excite quanta of the ϕ field (and its SUSY friends). Therefore, we can "integrate out" this field (this terminology comes from the Wilsonian idea of the RG: we take the path integral and integrate out modes of the fields above the RG scale... since the field has mass above the RG scale, we should integrate out all of its modes) by using its EOM to remove it from the theory. In particular, we can perform the F EOM from

$$\mathcal{L} \supset |F|^2 + 2Fm\phi + 2\bar{F}\bar{m}\bar{\phi} \ , \tag{1.11}$$

and obtain

$$\bar{F} = -2\phi \ , \tag{1.12}$$

so the condition of a SUSY vacuum is then that

$$0 = \bar{F} = -2m\phi \ . \tag{1.13}$$

We think of this as an operator equation that sets ϕ and all of its partners to zero. In particular, the theory in the IR is just the empty theory.

• Note also, in case it wasn't clear, similar comments apply to the fermion. So, our RG flow is a flow between the free massless chiral multiplet SCFT and the trivial theory in the IR.

• This discussion shows that the set of ideas behind renormalization really have nothing to do in general with computing loop diagrams. In some cases, we may need to compute loop diagrams in order to compute beta functions, but the idea and utility of the RG is much greater than these particular applications.

• In today's lecture, we would like to consider the more interesting deformation of the free theory

$$\delta W = m\Phi^2 + \lambda\Phi^3 \ . \tag{1.14}$$

When $\lambda \neq 0$, note that we have interactions (this term is also relevant since $R_0(\lambda) = \frac{1}{2} = \Delta(\lambda)$). For example, we have

$$\begin{aligned} \mathcal{L} &\supset -|W'|^2 + \frac{1}{2}W''\psi^2 - \frac{1}{2}W''\bar{\psi}^2 \supset -9|\lambda|^2|\phi|^4 - 6m\bar{\lambda}\phi\bar{\phi}^2 - 6\bar{m}\lambda\bar{\phi}\phi^2 + 3\lambda\psi^2\phi - 3\bar{\lambda}\bar{\psi}^2\bar{\phi} \\ &+ m\psi^2 - \bar{m}\bar{\psi}^2 \ . \end{aligned} \tag{1.15}$$

• To understand this theory and take quantum corrections into account, let's first step back and analyze a more general theory of chiral superfields, Φ_i

$$\mathcal{L} = -\int d^4\theta K(\Phi_i, \bar{\Phi}_i) + \left(\int d^2\theta W(\Phi_i) + \text{h.c.}\right) \ . \tag{1.16}$$

This is the most general two-derivative Lagrangian we can write. These are called (3D) Wess-Zumino (WZ) models (they have no gauge interactions, although we will see this distinction is, in some sense, a mirage). The second term is a holomorphic function of the chiral superfields and, as we know, is called the superpotential. The first term is a real function of Φ_i and $\bar{\Phi}_i$ called the "Kähler potential." It gives rise to a metric (called the Kähler metric)

$$g_{i\bar{j}}(\phi_k, \bar{\phi}_k) = \partial_i\partial_{\bar{j}}K(\phi_k, \bar{\phi}_k) \ , \tag{1.17}$$

where differentiation is with respect to the fields. This metric is invariant under a set of transformations called Kähler transformations $K(\phi_i, \bar{\phi}_i) \to K + \chi(\phi_i) + \bar{\chi}(\bar{\phi}_i)$. Note that since we integrate these deformations over all of superspace, they are total derivatives and therefore do not affect the theory.

• Unitarity of the theory (right sign kinetic term) requires that $g_{i\bar{j}} \geq 0$ (with vanishing norm only for a zero vector). We have

$$\begin{aligned} \mathcal{L} &= g_{i\bar{j}}\left(-\partial_\mu\phi^i\partial^\mu\bar{\phi}^j + F^i\bar{F}^{\bar{j}} + i\psi^i\gamma^\mu\partial_\mu\bar{\psi}^{\bar{j}}\right) + \frac{1}{4}R_{i\bar{j}k\bar{l}}\psi^i\psi^k\bar{\psi}^{\bar{j}}\bar{\psi}^{\bar{l}} + F^iW_i + \bar{F}^{\bar{i}}\bar{W}_{\bar{i}} \\ &+ \frac{1}{2}W_{ij}\psi^i\psi^j - \frac{1}{2}\bar{W}_{\bar{i}\bar{j}}\bar{\psi}^{\bar{i}}\bar{\psi}^{\bar{j}} \ . \end{aligned} \tag{1.18}$$

The F-term equations are

$$g_{i\bar{j}}\bar{F}^{\bar{j}} + W_i = 0 \ \Rightarrow \ \bar{F}^{\bar{j}} = -g^{i\bar{j}}W_i \ . \tag{1.19}$$

So we get

$$
\begin{aligned}
\mathcal{L} &= g_{i\bar{j}}\left(-\partial_\mu\phi^i\partial^\mu\bar{\phi}^j + i\psi^i\gamma^\mu\partial_\mu\bar{\psi}^{\bar{j}}\right) + \frac{1}{4}R_{i\bar{j}k\bar{l}}\psi^i\psi^k\bar{\psi}^{\bar{j}}\bar{\psi}^{\bar{l}} - g_{i\bar{j}}W^i\bar{W}^{\bar{j}} \\
&+ \frac{1}{2}W_{ij}\psi^i\psi^j - \frac{1}{2}\bar{W}_{\bar{i}\bar{j}}\bar{\psi}^{\bar{i}}\bar{\psi}^{\bar{j}} \ .
\end{aligned}
\tag{1.20}
$$

The classical SUSY vacua are at $W^i = 0$ (since $g^{i\bar{j}} \geq 0$).

• Recall that in SQM, we had classical SUSY vacua described by the vanishing of the auxiliary fields

$$
\bar{F}^{\bar{j}} = -g^{i\bar{j}}W_i = 0 \ , \tag{1.21}
$$

as well. However, non-perturbative tunneling effects (via instantons) could lift these vacua and make them non-SUSY in the quantum theory [**Draw $V = (W')^2$ with two minima**]. As you know, tunneling events are exponentially suppressed, $e^{-\frac{c}{\hbar}}$.

• In QFT, in the infinite volume limit, tunneling between vacua is completely suppressed, $e^{-\frac{c}{\hbar}\mathrm{Vol(Space)}} \to 0$ as Vol(Space) $\to \infty$...

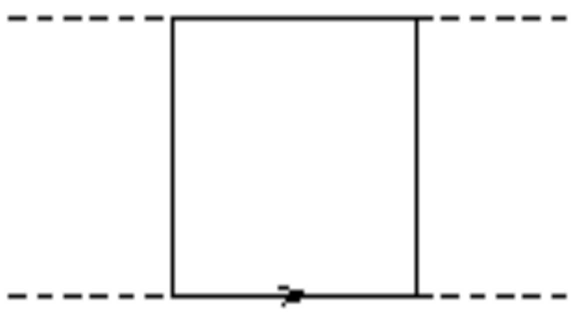

Fig. 1: The generation of a ϕ^4 term in Yukawa theory.

• However, to know which vacua we have (a low energy question), we need to understand how the theory behaves at low energies. In general in QFT this is a difficult thing to do. The main reason is renormalization. We define the theory in the UV. Even if we start with a set of interactions where the theory is defined in the UV, we get divergences and need to introduce new interaction terms that cancel these interactions. For example, in (non-SUSY) Yukawa theory, say in 4D, we have the situation in Fig. 1, where we start with $\mathcal{L} \supset h\psi\psi\phi$ (I am using two-component spinors) and generate $\lambda\phi^4$. Since this term is log-divergent, we add a term to the action that *cancels* this divergence (so that the low-energy theory is independent of the cut-off).

• This term is called a "counterterm," and is a function of μ in our way of performing renormalization (this is also the scale at which we observe the theory). Differentiating the coupling with respect to $\log \mu$ gives us the beta function. The idea here is to add terms to the action that keep the low-energy theory fixed. In particular, we find something like $\beta_\lambda = \frac{1}{16\pi^2}(3\lambda^2 + 8\lambda h^2 - 48h^4) + \cdots$... This is the Wilsonian approach to the RG...

• What happens with SUSY? Well, let's go back to the theories of chiral superfields, Φ_i

$$\mathcal{L} = -\int d^4\theta \bar{\Phi}\Phi + \left(\int d^2\theta (m\Phi^2 + \lambda\Phi^3) + \text{h.c.}\right) . \tag{1.22}$$

For simplicity let us first set $m = 0$.

• Then, $\delta W = \lambda\Phi^3$, and, although the theory is non-conformal ($R_0(\Phi^3) = \frac{3}{2}$, so $R_0(\lambda) = \frac{1}{2}$), it has an R symmetry, $R = R_{\frac{1}{6}} = R_0 + \frac{1}{6}\mathcal{J}$ (recall $\mathcal{J}(\phi) = \mathcal{J}(\psi) = \mathcal{J}(F) = +1$) under which $R(\phi) = \frac{2}{3}$, $R(\psi) = -\frac{1}{3}$, $R(F) = -\frac{4}{3}$.

• Let us now think of λ as a chiral superfield spurion / background field that transforms under $\mathcal{J}$. *(Note: this is the promised higher-dimensional version of the discussion of background fields we first encountered in* $(2,2)$ *Berry phase in* $0+1$ *dimensions.)* Then, we need $\mathcal{J}(\lambda) = -3$ to make this a "symmetry" (we also have $R(\lambda) = 0$). What kinds of terms can appear in the superpotential after turning on this coupling? Well, they must be of the form $y^p\Phi^q \subset W$. Indeed, SUSY doesn't allow $\bar{y}$ or $\bar{\Phi}$ to appear. Now, invariance under the R symmetry requires $q = 3$. On the other hand, invariance under $\mathcal{J}$ requires $p = 1$. Therefore, we conclude that W is not renormalized at all! Similar logic holds if we had taken $\delta W = \lambda_k \Phi^k$ (**note:** with no summation over k). This statement holds perturbatively (i.e., for polynomials in y) and non-perturbatively in y (i.e., non-polynomially in y)!

• Note, however, that the Kähler potential can get renormalized since both y and $\bar{y}$ are allowed to enter (**Exercise:** although not at one loop! Mention Feynman diagrams...). But, since we started with a unitary theory (probabilities are conserved), we must end up with a unitary theory. Therefore, we have a unique SUSY vacuum (we say there is NO moduli space) at $\bar{F} = -3g^{1\bar{1}}\lambda X^2 = 0$ (where $g_{1\bar{1}} > 0$ is the Kähler metric and depends on $y, \bar{y}$), so $\langle X \rangle = 0$.... Note that X is not massive, so we do not integrate it out!

• Instead, since there is no mass, it is reasonable to believe there is an SCFT in the IR (at zero energy, we must either get a massive theory or a theory without scale...). What is the scaling dimension of Φ in the IR? Well, we saw in our previous lecture that there **must** be a superconformal R symmetry in such an SCFT, R_0. If we **assume** there are no accidental flavor symmetries in the IR (this is what's usually done unless we violate certain unitarity bounds... we won't have time to get to this in our module, so we will always assume unitarity bounds are not violated), we must have

$$R_0^{IR}(\phi) = \frac{2}{3} = \Delta_{IR}(\phi) \ . \tag{1.23}$$

Note

$$\Delta_{UV} - \Delta_{IR}(\phi) = \frac{1}{2} - \frac{2}{3} = -\frac{1}{6} \tag{1.24}$$

In particular, the scaling dimension of ϕ increases by 1/6! This implies we have strong quantum corrections in the IR!!! Try to compute this from Feynman diagrams, it is impossible. Note that this cannot be a free SCFT. Such a theory necessarily has free chiral superfields (free vectors are actually not conformal in 3D!!!). In particular, in a free theory, scaling dimensions of all operators must be integer or half-integer...

• Note that you may also have worried about higher-derivative terms, e.g., $\int d^4\theta \bar{D}\bar{\Phi}D\Phi$... However, these terms necessarily come with more derivatives, so they vanish in the limit of zero momentum (i.e., they are always irrelevant in the IR...).

• Next let us consider turning $m \neq 0$ as well so that $\delta W = m\Phi^2 + \lambda\Phi^3$... Now we have a spurious R symmetry under which $R(m) = \frac{2}{3}$ (the rest of the fields still transform in the original way and $R(\lambda) = 0$) and a spurious $\mathcal{J}$ symmetry under which $\mathcal{J}(m) = -2$.

• Therefore, the quantum superpotential must have the form

$$W = m\Phi^2 \cdot f\left(\frac{\lambda\Phi}{m}\right) \ , \tag{1.25}$$

where $\mathcal{J}(\lambda\phi/m) = \mathcal{J}(\lambda) + \mathcal{J}(\phi) - \mathcal{J}(m) = -3 + 1 + 2 = 0$ and $R(\lambda\phi/m) = R(\lambda) + R(\phi) - R(m) = 0 + \frac{2}{3} - \frac{2}{3} = 0$ while $\mathcal{J}(m\Phi^2) = 2$ and $R(m\Phi^2) = 2$.

• Denote $u = \lambda\Phi/m$.... In particular, holomorphy of the superpotential means f is a holomorphic function of u. Clearly at $\lambda = 0$ with $m \neq 0$, $u = 0$. We know that if the interaction is turned off, then

$$f(0) = 1 \ . \tag{1.26}$$

we also saw that when $m \to 0$ with $\lambda \neq 0$, we have

$$f(|u| \to \infty) = u \ . \tag{1.27}$$

What about higher-order terms? For example, a Φ^4 term would look like $\frac{\lambda^2}{m}\Phi^4 \subset W$. However, it is easy to check that loops that can contribute to, say, $\phi^2\psi^2$ start at least at order λ^4 (really $\lambda^3\lambda^*$)....

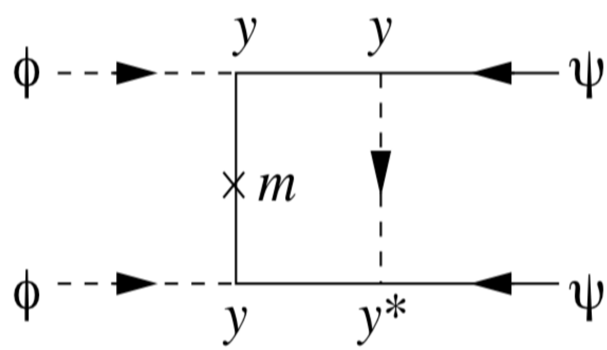

Fig. 2: The generation of a $\phi^2\psi^2$ term (here $y \to \lambda$) starts at order too high in λ.

• Therefore, there are no such terms in W... Similarly for higher orders as well... This fixes the superpotential... Even including non-perturbative effects. For example

$$e^{-\frac{1}{u^2}} + 1 + u \ , \tag{1.28}$$

has the correct behavior as $u \to \infty$ and as $u \to 0$ if $u \in \mathbb{R}$... However, if $u \in i\mathbb{R}$, then we get the wrong behavior as $u \to 0$... The correct answer therefore is

$$f(u) = 1 + u \ . \tag{1.29}$$

The superpotential is not renormalized.

• The above non-renormalization arguments extend quite generally to WZ models and also to SQED... There is renormalization, but it is through the Kähler potential (note that the gauge kinetic terms, which do get renormalized, can always be written as Kähler potential terms...).

• We can also have interesting examples with moduli spaces (i.e., spaces of vacua). For example, consider the following

$$\mathcal{L} = -\int d^4\theta(\bar{X}X + \bar{Y}Y + \bar{Z}Z) + \left(\int d^2\theta\lambda XYZ + \text{h.c.}\right) \ . \tag{1.30}$$

We have the following set of vacua

$$\bar{F}_X = -\lambda YZ = 0 \ , \quad \bar{F}_Y = -\lambda XZ = 0 \ , \quad \bar{F}_Z = -\lambda XY = 0 \ . \tag{1.31}$$

This has the following three solutions

$$\langle X \rangle \ = \ \langle Y \rangle = 0 \ , \quad \langle Z \rangle \in \mathbb{C} \ ,$$

$$
\begin{aligned}
\langle X\rangle &= \langle Z\rangle = 0\ , \quad \langle Y\rangle \in \mathbb{C}\ , \\
\langle Y\rangle &= \langle Z\rangle = 0\ , \quad \langle X\rangle \in \mathbb{C}\ .
\end{aligned} \tag{1.32}
$$

These are three "branches..." Each parameterized by vevs for X, Y, and Z... They meet at the point where $\langle X\rangle = \langle Y\rangle = \langle Z\rangle = 0$ is an interacting SCFT. This is called the "origin of the moduli space." On each of these branches, we have massless modes... As we will see, these are related to spontaneously broken symmetries (i.e., symmetries preserved by the theory but broken by the vacuum)... These are called Goldstone bosons (and their super-partners)... For example, on each branch, we break a $U(1)_R$ under which $R(X) = R(Y) = R(Z) = \frac{2}{3}$, so we have a multiplet corresponding to this broken generator...

• Next week we will delve into Goldstone's theorem, the super Higgs mechanism, and SQED's moduli space...

References

[1] A. B. Zamolodchikov, *"Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory"*, JETP Lett. **43**, 730 (1986), [Pisma Zh. Eksp. Teor. Fiz.43,565(1986)]✦ H. Casini & M. Huerta, *"On the RG running of the entanglement entropy of a circle"*, Phys. Rev. **D85**, 125016 (2012), `arXiv:1202.5650`✦ Z. Komargodski & A. Schwimmer, *"On Renormalization Group Flows in Four Dimensions"*, JHEP **1112**, 099 (2011), `arXiv:1107.3987`