SPA5304 Physical Dynamics Lecture 25

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1 Hamiltonian mechanics

1.1 Example: Harmonic oscillator

$$L = \frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

We need to express $\dot{x} = \dot{x}(x, p, t)$. We get

$$\dot{x} = \frac{p}{m}$$

$$\boxed{H = p\dot{x} - L = \frac{p^2}{2m} + \frac{1}{2}kx^2}$$

Hamilton's equations:

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p} \\ \\ \dot{p} = -\frac{\partial H}{\partial x} \end{cases} \qquad \Rightarrow \qquad \begin{cases} \dot{x} = \frac{p}{m} \\ \\ \dot{p} = -kx \end{cases}$$

This can be converted into a second order equation by plugging p into the second equation:

$$m\ddot{x} = -kx$$

We get the good old Newton's equation.

$$\ddot{x} + \omega^2 x = 0, \qquad \omega \equiv \sqrt{\frac{k}{m}}$$

The solution:

$$x = A\cos(\omega t + \alpha)$$

$$p = m\dot{x} = -Am\omega\sin(\omega t + \alpha)$$

As a trajectory in phase space:

$$\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} A\cos(\omega t + \alpha) \\ -Am\omega\sin(\omega t + \alpha) \end{pmatrix}$$

The energy:

$$E = H = \frac{p^2}{2m} + \frac{1}{2}kx^2 = \frac{k}{2}A^2$$

This equation describes an ellipse in phase space.

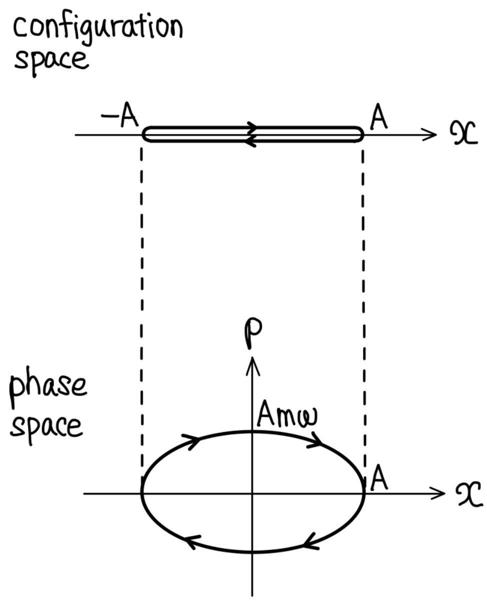
Trajectories are parametrized by A.

Motion in configuration space:

- The particle moves back and forth between A and $-A$.
- Trajectories with different values of A overlap.
- $x(0)$ is not enough to determine $x(t)$ for $t > 0$.

Motion in phase space:

- A trajectory is an ellipse.
- Trajectories with different values of A do not overlap.
- giving an initial position $(x(0), p(0))$ is enough to determine the time-evolution.



1.2 Example: A particle in a potential

$$L = \frac{m}{2}\dot{x}^2 - V(x)$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

We need to express $\dot{x} = \dot{x}(x, p, t)$. We again get

$$\dot{x} = \frac{p}{m}$$

$$H = p\dot{x} - L = \frac{p^2}{m} - \left[\frac{m}{2}\left(\frac{p}{m}\right)^2 - V(x)\right] = \frac{p^2}{2m} + V(x)$$

Thus,

$$\boxed{H = \frac{p^2}{2m} + V(x)}$$

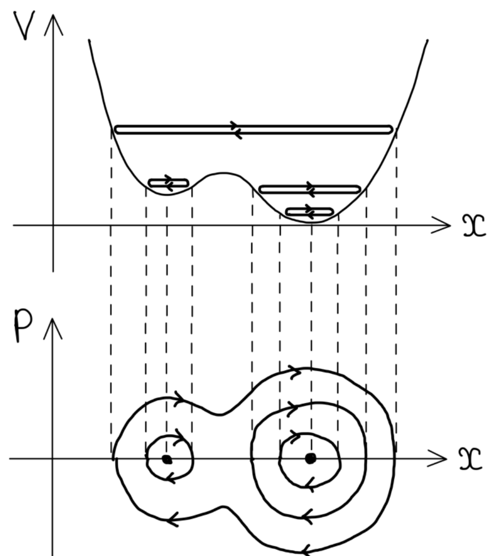
Hamilton's equations:

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \\ \\ \dot{p} = -\frac{\partial H}{\partial x} = -V'(x) \end{cases} \qquad \Rightarrow \qquad m\ddot{x} = -V'(x)$$

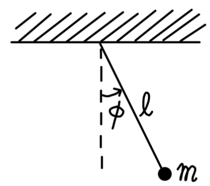
- Motion in configuration space, as read off from the potential

- Motion in phase space.

  $(x(0), p(0))$ is enough to determine $(x(t), p(t))$, $t > 0$.



1.3 Example: A planar pendulum



$$L(\phi,\dot\phi)=\frac{m}{2}l^2\dot\phi^2+mgl\cos\phi$$

$$p_\phi\equiv\frac{\partial L}{\partial\dot\phi}=ml^2\dot\phi$$

We need to express $\dot\phi$

$$\dot\phi=\frac{p_\phi}{ml^2}$$

$$H=p_\phi\dot\phi-L=p_\phi\frac{p_\phi}{ml^2}-\frac{1}{2}ml^2\left(\frac{p_\phi}{ml^2}\right)^2-mgl\cos\phi=\frac{p_\phi}{2ml^2}-mgl\cos\phi$$

$$\boxed{H=\frac{p_\phi}{2ml^2}-mgl\cos\phi}$$

Hamilton's equations:

$$\begin{cases}\dot\phi=\frac{\partial H}{\partial p_\phi}=\frac{p_\phi}{ml^2}\\ \\ \dot{p_\phi}=-\frac{\partial H}{\partial\phi}=-mgl\sin\phi\end{cases}$$

Thus,

$$ml^2\ddot\phi=-mgl\sin\phi$$

or

$$\ddot\phi=-\frac{g}{l}\sin\phi$$

1.4 Many degrees of freedom

Straightforward generalization of the 1 DoF case.

Start with $L(\vec{q}, \dot{\vec{q}}, t)$, where $\vec{q} = (q_1, \ldots, q_n)$.

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \qquad \Rightarrow \qquad \dot{q}_i = \dot{q}_i(\vec{q}, \vec{p}, t)$$

Then the Hamiltonian is

$$\boxed{H(\vec{q}, \vec{p}, t) = p_i \dot{q}_i - L}$$

where a summation over $i = 1, \ldots, n$ is understood.

- H is again the energy expressed as a function of $\vec{q}, \vec{p}, t$ instead of $\vec{q}, \dot{\vec{q}}, t$.
- Hamilton's equations can be derived exactly the same way (just add the index i):

$$\boxed{\dot{q}_i = \frac{\partial H}{\partial p_i}}$$

and

$$\boxed{\dot{p}_i = -\frac{\partial H}{\partial q_i}}$$

and we also have

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

- Hamilton's equations are a system of first-order differential equations for $2n$ variables $\vec{q}, \vec{p}$.

1.5 Cyclic coordinates

Let us recall that we called generalized coordinates cyclic if they did not appear in the Lagrangian:

$$\frac{\partial L}{\partial q_i} = 0$$

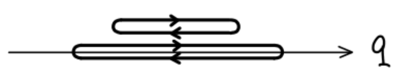
In this case the conjugate momentum is conserved:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \text{const.}$$

In the Hamiltonian formulation,

$$\frac{\partial H}{\partial \dot{q}_i} = -\dot{p}_i = -\frac{\partial L}{\partial q_i}$$

so if q_i does not appear in L, then it will not appear in H either, and vice versa.

Lagrangian	Hamiltonian
E-L eqs: $\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{q}}} = \frac{\partial L}{\partial \vec{q}}$ Second-order DEs for n variables $\vec{q}$	Hamilton's eqs.: $\dot{\vec{q}} = \frac{\partial H}{\partial \vec{p}}$, $\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{q}}$ First-order DEs for $2n$ variables $(\vec{q}, \vec{p})$
The n-dimensional space is called "configuration space"	The $2n$-dimensional space is called "phase space"
$\vec{q}(0)$ is not enough to determine $\vec{q}(t)$, $t>0$	$(\vec{q}(0), \vec{p}(0))$ is enough to determine $(\vec{q}(t), \vec{p}(t))$, $t>0$
Different trajectories can overlap. 	Different trajectories do not intersect with each other