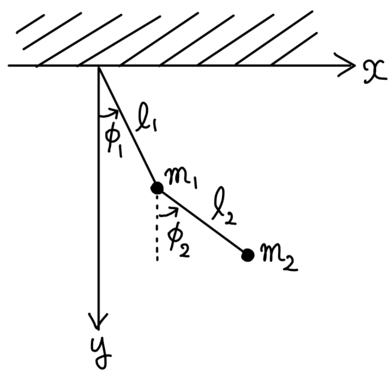
SPA5304 Physical Dynamics Lecture 23-24

David Vegh
(figures by Masaki Shigemori)

6 March 2019

1 Small Oscillations

1.1 Example: Double pendulum



- 2 DoFs

$$\begin{cases} x_1 = l_1 \sin\phi_1 \\ y_1 = l_1 \cos\phi_1 \end{cases}$$

$$\begin{cases} x_2 = x_1 + l_2 \sin\phi_2 \\ y_2 = y_1 + l_2 \cos\phi_2 \end{cases}$$

$$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) = \frac{1}{2}m_1 l_1^2 \dot{\phi}_1^2 + \frac{1}{2}m_2(l_1^2\dot{\phi}_1^2 + l_2^2\dot{\phi}_2^2 + 2l_1 l_2 \dot{\phi}_1\dot{\phi}_2 \cos(\phi_1 - \phi_2))$$

$$V = -m_1 g y_1 - m_2 g y_2 = -g(l_1(m_1 + m_2)\cos\phi_1 + l_2 m_2 \cos\phi_2)$$

For simplicity, let us set

$$m_1 = m_2 = m, \qquad l_1 = l_2 = l$$

Then,

$$T = ml^2\dot{\phi}_1^2 + \frac{1}{2}ml^2\dot{\phi}_2^2 + ml^2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2)$$
$$V = -2mgl\cos\phi_1 - mgl\cos\phi_2$$

- Equilibrium positions are found by solving

$$\left.\begin{array}{l} \frac{\partial V}{\partial \phi_1} = 2mgl\sin\phi_1 \\ \\ \frac{\partial V}{\partial \phi_2} = mgl\sin\phi_2 \end{array}\right\} \quad \Rightarrow \quad \phi_1 = \phi_2 = 0$$

Other solutions, such as $\phi_1 = \phi_2 = \pi$ are clearly unstable.

- The Hessian

$$\left.\begin{array}{l} \frac{\partial^2 V}{\partial \phi_1^2} = 2mgl\cos\phi_1 \\ \\ \frac{\partial^2 V}{\partial \phi_2^2} = mgl\cos\phi_2 \\ \\ \frac{\partial^2 V}{\partial \phi_1 \partial \phi_2} = 0 \end{array}\right\} \quad \Rightarrow \quad \mathcal{V}_{ij} = mgl\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

- Kinetic energy

$$T = ml^2\dot{\phi}_1^2 + \frac{1}{2}ml^2\dot{\phi}_2^2 + ml^2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2) \equiv \frac{1}{2}a_{ij}(\phi_1, \phi_2)\dot{\phi}_i\dot{\phi}_j$$

From this we can read off a_{ij}:

$$a_{ij} = \begin{pmatrix} 2ml^2 & ml^2\cos(\phi_1 - \phi_2) \\ ml^2\cos(\phi_1 - \phi_2) & ml^2 \end{pmatrix}$$

and finally,

$$\mathcal{T}_{ij} = a_{ij}(\phi_1 = 0,\, \phi_2 = 0) = ml^2\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Note: be careful with the $\frac{1}{2}$ when reading off a_{ij}!

- Let us now follow the procedure. The secular equation:

$$\det(\omega^2\mathcal{T} - \mathcal{V}) = 0$$

$$\det\left[\omega^2 ml^2\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} - mgl\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\right] = 0$$

$$\det\begin{pmatrix} 2(\omega^2 - g/l) & \omega^2 \\ \omega^2 & \omega^2 - g/l \end{pmatrix} = 0$$

$$2(\omega^2 - g/l)^2 - \omega^4 = 0$$

$$\omega^4 - 4\frac{g}{l}\omega^2 + 2\frac{g^2}{l^2} = 0$$

$$\omega^2 = \frac{g}{l}(2 \pm \sqrt{2})$$

So the normal frequencies are

$$\omega_{(1)}^2 = \frac{g}{l}(2-\sqrt{2})$$
$$\omega_{(2)}^2 = \frac{g}{l}(2+\sqrt{2})$$

- Plug these back in to find amplitudes A

$$(\omega^2 \mathcal{T}_{lj} - \mathcal{V}_{lj})A_j = 0$$

$$\begin{pmatrix} 2(\omega^2 - g/l) & \omega^2 \\ \omega^2 & \omega^2 - g/l \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

Since the determinant of the matrix vanishes, the two equations are not independent. We only need to solve one of them.

- Plug in $\omega_{(1)}^2$

$$2\left(\frac{g}{l}(2-\sqrt{2}) - \frac{g}{l}\right)A_1 + \frac{g}{l}(2-\sqrt{2})A_2 = 0$$
$$\sqrt{2}A_1 - A_2 = 0$$
$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = c\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \qquad c \in \mathbb{C}$$

This is the first normal mode that we have found.

Solution

$$\begin{cases} \eta_1 = Re[ce^{i\omega_{(1)}t}] \\ \eta_2 = Re[\sqrt{2}ce^{i\omega_{(1)}t}] \end{cases}$$

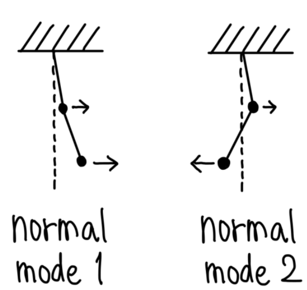
- Plug in $\omega_{(2)}^2$

$$2\left(\frac{g}{l}(2+\sqrt{2}) - \frac{g}{l}\right)A_1 + \frac{g}{l}(2+\sqrt{2})A_2 = 0$$
$$\sqrt{2}A_1 + A_2 = 0$$
$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = c\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}, \qquad c \in \mathbb{C}$$

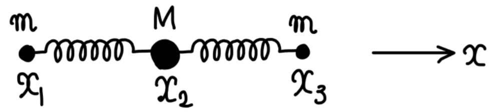
This is the second normal mode that we have found.

Solution

$$\begin{cases} \eta_1 = Re[ce^{i\omega_{(2)}t}] \\ \eta_2 = Re[-\sqrt{2}ce^{i\omega_{(2)}t}] \end{cases}$$



1.2 Example: Linear triatomic molecule in 1D



- 3 DoFs
- Modeled with spring forces (e.g. CO_2).

$$\begin{cases} T = & \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_3^2) + \frac{1}{2}M\dot{x}_2^2 \\ V = & \frac{1}{2}k(x_2 - x_1 - l)^2 + \frac{1}{2}k(x_3 - x_2 - l)^2 \end{cases}$$

where l is the rest length of the springs.

Equilibrium positions are found by solving

$$\begin{cases} 0 = \frac{\partial V}{\partial x_1} = & -k(x_2 - x_1 - l) \\ \\ 0 = \frac{\partial V}{\partial x_2} = & k(x_2 - x_1 - l) - k(x_3 - x_2 - l) = k(2x_2 - x_1 - x_3) \\ \\ 0 = \frac{\partial V}{\partial x_3} = & k(x_3 - x_2 - l) \end{cases}$$

Thus,

$$x_2 - x_1 = l, \qquad x_3 - x_2 = l$$

Note that only relative positions are determined.

- Hessian:

$$\frac{\partial^2 V}{\partial x_1^2} = k, \qquad \frac{\partial^2 V}{\partial x_2^2} = 2k, \qquad \frac{\partial^2 V}{\partial x_3^2} = k$$

$$\frac{\partial^2 V}{\partial x_1 \partial x_2} = -k, \qquad \frac{\partial^2 V}{\partial x_1 \partial x_3} = 0, \qquad \frac{\partial^2 V}{\partial x_2 \partial x_3} = -k.$$

Thus,

$$\mathcal{V}_{ij} = k \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

The matrix is independent of the x's.

$$\mathcal{T}_{ij} = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}$$

- Let us now again follow the procedure. The secular equation:

$$\det(\omega^2 \mathcal{T} - \mathcal{V}) = 0$$

$$\det \begin{pmatrix} m\omega^2 - k & k & 0 \\ k & M\omega^2 - 2k & k \\ 0 & k & m\omega^2 - k \end{pmatrix} = 0$$

$$(m\omega^2 - k)^2(M\omega^2 - 2k) - 2k^2(m\omega^2 - k) = 0$$

- Solutions:

$$\omega^2_{(0)} = 0$$

$$\omega^2_{(1)} = \frac{k}{m}$$

$$\omega^2_{(2)} = \frac{k}{m}\left(1 + \frac{2m}{M}\right)$$

- The amplitudes:

$$\begin{pmatrix} m\omega^2 - k & k & 0 \\ k & M\omega^2 - 2k & k \\ 0 & k & m\omega^2 - k \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0$$

- Plug in $\omega^2_{(0)} = 0$

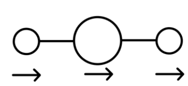
$$k \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0$$

Thus,

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\eta_1 = \eta_2 = \eta_3 = Re[ce^{i \times 0 \times t}] = \text{const.}$$

Overall translation



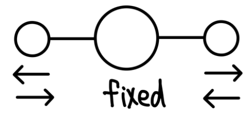
- Plug in $\omega^2_{(1)} = \frac{k}{m}$

$$k \begin{pmatrix} 0 & 1 & 0 \\ 1 & \frac{M}{m} - 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0$$

Thus,

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = c \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\eta_1 = -\eta_3 = Re[ce^{i\sqrt{\frac{k}{m}}t}], \qquad \eta_2 = 0$$



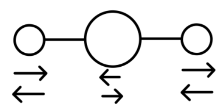
- Plug in $\omega_{(1)}^2 = \frac{k}{m}\left(1+\frac{2m}{M}\right)$

$$\begin{pmatrix} k\left(1+\frac{2m}{M}\right)-k & k & 0 \\ k & M\frac{k}{m}\left(1+\frac{2m}{M}\right) & k \\ 0 & k & k\left(1+\frac{2m}{M}\right)-k \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0$$

Thus,

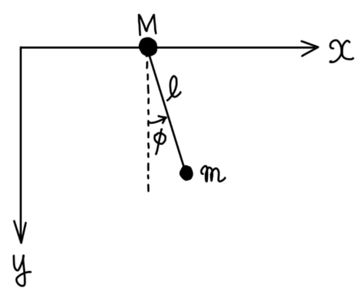
$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = c\begin{pmatrix} 1 \\ -\frac{2m}{M} \\ 1 \end{pmatrix}$$

$$\eta_1 = \eta_3 = Re[ce^{i\sqrt{\frac{k}{m}\left(1+\frac{2m}{M}\right)}t}], \qquad \eta_2 = -\frac{2m}{M}\eta_1$$



If $m \ll M$, then the middle one doesn't move.

1.3 Example: Pendulum with moving suspension point



- 2 DoFs

$$\begin{cases} x_M = x \\ y_M = 0 \end{cases} \qquad \begin{cases} x_m = x + l\sin\phi \\ y_m = l\cos\phi \end{cases}$$

$$\begin{cases} T = & \frac{1}{2}M(\dot{x}_M^2+\dot{y}_M^2)+\frac{1}{2}m(\dot{x}_m^2+\dot{y}_m^2) = \frac{M}{2}\dot{x}^2+\frac{m}{2}(\dot{x}^2+l^2\dot{\phi}^2+2l\dot{x}\dot{\phi}\cos\phi) \\ V = & -mgy_m = -mgl\cos\phi \end{cases}$$

- Equilibrium positions

$$\frac{\partial V}{\partial x} = 0, \qquad \frac{\partial V}{\partial \phi} = mgl\sin\phi = 0$$

This gives $\phi = 0$ and $x =$arbitrary.

- The $\mathcal{T}$ and $\mathcal{V}$ matrices:

$$\mathcal{T} = \begin{pmatrix} M+m & ml \\ ml & ml^2 \end{pmatrix}$$

$$\mathcal{V} = \begin{pmatrix} 0 & 0 \\ 0 & mgl \end{pmatrix}$$

- secular equation

$$\det \begin{pmatrix} \omega^2(M+m) & \omega^2 ml \\ \omega^2 ml & \omega^2 ml^2 - mgl \end{pmatrix} = 0$$

- The solutions are

$$\omega^2_{(0)} = 0$$

This is again an overall translation.

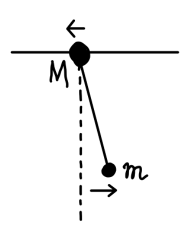
$$\omega^2_{(1)} = \frac{g}{l}\left(1 + \frac{m}{M}\right)$$

$$\begin{pmatrix} \omega^2(M+m) & \omega^2 ml \\ \omega^2 ml & \omega^2 ml^2 - mgl \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$$A_1 = -\frac{ml}{M+m} A_2$$

$$x(t) = x_{\text{arbitrary}} - \frac{ml}{M+m} Re\left[A_2 e^{i\sqrt{\frac{g}{l}\left(1+\frac{m}{M}\right)}t}\right]$$

$$\phi(t) = Re\left[A_2 e^{i\sqrt{\frac{g}{l}\left(1+\frac{m}{M}\right)}t}\right]$$



If $m \ll M$, then $x = x_{\text{arbitrary}}$: the problem reduces to a simple pendulum.

2 Hamiltonian mechanics

So far w ehave been using the Lagrangian formalism of mechanics, in which the system is described by generalized coordinates $\vec{q}$ and generalized velocities $\dot{\vec{q}}$.

We now want to introduce the Hamiltonian formalism, in which the system is described by generalized coordinates $\vec{q}$ and generalized momenta $\vec{p}$.

This formalism is even more general: it treats $\vec{q}$ and $\vec{p}$ on equal footing.

2.1 Legendre transformation

- It is a procedure to go from $(q, \dot{q})$ to (q, p).
- Very often used in thermodynamics.

Consider one degree of freedom. Let us eliminate $\dot{q}$ in terms of $p \equiv \frac{\partial L}{\partial \dot{q}}$. This gives a function

$$\boxed{\dot{q} = \dot{q}(q, p, t)}$$

Now go from the Lagrangian $L(q, \dot{q}, t)$ to the **Hamiltonian** $H(q, p, t)$ by

$$\boxed{H = p\dot{q} - L}$$

The right hand side involves $\dot{q}$ which we need to reexpress in terms of p. Then we get $H = H(q, p, t)$.

2.2 Hamilton's equations

Consider a small change in H

$$dH = d(p\dot{q} - L) = dp\dot{q} + pd\dot{q} - dL$$

Here

$$dL = \frac{\partial L}{\partial q}dq + \frac{\partial L}{\partial \dot{q}}d\dot{q} + \frac{\partial L}{\partial t}dt$$

So

$$dH = dp\dot{q} + pd\dot{q} - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial \dot{q}}d\dot{q} - \frac{\partial L}{\partial t}dt = \dot{q}dp + \underbrace{\left(p - \frac{\partial L}{\partial \dot{q}}\right)}_{0} d\dot{q} - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial t}dt$$

Thus,

$$dH = \dot{q}dp - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial t}dt$$

On the other hand, $H = H(q, p, t)$ gives

$$dH = \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial p}dp + \frac{\partial H}{\partial t}dt$$

Since q, p, t are independent, comparing the coefficients gives

$$\frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q}, \qquad \frac{\partial H}{\partial p} = \dot{q}, \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

These simply followed from the definition of p and H.

Let us now also use the Euler-Lagrange equation $\dot{p} = \frac{\partial L}{\partial q}$ to get **Hamilton's equations**

$$\boxed{\dot{q} = \frac{\partial H}{\partial p}}$$

and

$$\boxed{\dot{p} = -\frac{\partial H}{\partial q}}$$

• Note that q and p appear symmetrically (up to a sign).

• These are first-order differential equations for two variables, in contrast to the Euler-Lagrange equation which is a second-order equation for a single variable q.

• The 2d space of (q, p) is called the **phase space**. (Previously we have defined the configuration space which is the 1d space of q.)

• The Hamiltonian is the energy of the system: the Noether invariant associated to time-translation.

• Given the initial position (q, p) in pahse space at $t = t_0$, $H(q, p)$ tells us how to evolve in t:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{pmatrix}$$

For instance, if $H = \frac{1}{2}(p^2 + q^2)$, then

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix}$$