SPA5304 Physical Dynamics Lecture 22

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1 Small Oscillations

1.1 Summary

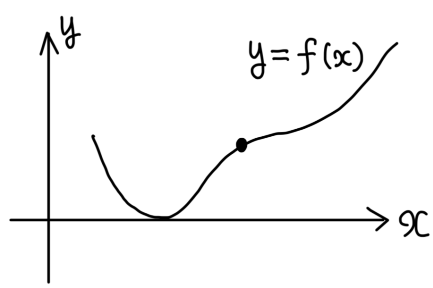
- $\dot{q} = 0$, $q = q_0$ is a stable equilibrium position iff $V'(q_0) = 0$ and $V''(q_0) > 0$.
- In this case, if we write $q = q_0 + \eta$ where η is small, then η will do small harmonic oscillations.
- The frequency of oscillations is

$$\omega = \sqrt{\frac{V''(q_0)}{a(q_0)}}$$

- The period is $T = \frac{2\pi}{\omega}$.
- If $V'(q_0) = 0$ but $V''(q_0) < 0$, then $q = q_0$ is an unstable equilibrium position. In this case

$$\omega^2 \equiv \frac{V''(q_0)}{a(q_0)} < 0$$

1.2 Example: A bead on a wire of equation $y = f(x)$



- #DoF = 1
- generalized coordinate: x
- potential: $V = mgy = mgf(x)$
- kinetic energy:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(\dot{x}^2 + f'(x)^2\dot{x}^2) = \frac{1}{2}m\dot{x}^2(1 + f'(x)^2)$$

- Lagrangian:

$$L = T - V = \frac{1}{2}m\dot{x}^2(1 + f'(x)^2) - mgf(x)$$

- Equilibrium position:

$$\begin{cases} V'(x_0) = 0 \\ V''(x_0) > 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} f'(x_0) = 0 \\ f''(x_0) > 0 \end{cases}$$

Namely, x_0 must be a local minimum of $f(x)$.

- frequency of small oscillations:

  Expand $x = x_0 + \eta + \mathcal{O}(\eta^2)$ and use $f'(x_0) = 0$

$$T = \frac{1}{2}m\dot{x}^2(1 + f'(x)^2) \to \frac{1}{2}m\dot{x}^2(1 + \underbrace{f'(x_0)^2}_{0}) = \frac{1}{2}m\dot{\eta}^2$$

$$V \to \frac{1}{2}mf''(x_0)\eta^2$$

$$\omega^2 = \frac{V''(q_0)}{a(q_0)} = \frac{mgf''(x_0)}{m}$$

$$\omega = \sqrt{gf''(x_0)}$$

1.3 Many variables

$$L = \frac{1}{2}a_{ij}(\vec{q})\dot{q}_i\dot{q}_j - V(\vec{q})$$

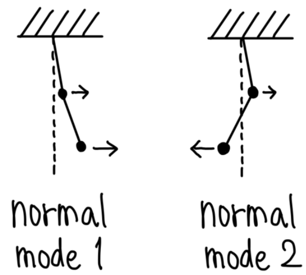
where $i, j = 1, \ldots, n$.

- We will study equilibrium positions, just as in the 1 DoF case.
- We will also study small fluctuations around stable equilibrium positions.

1.4 New features

- Small fluctuations behave as independent harmonic oscillations.
- The motion of the system can be thought of as a superposition of **normal modes**.
- A normal mode is a pattern of motion in which all the degrees of freedom in the system oscillate at the same frequency, called the **normal frequency**.
- In an n DoF system, there are n normal modes.

For instance: Double pendulum (#DoF = 2)



1.5 Stability

As in the 1 DoF case, we define a stable equilibrium position

$$\dot{\vec{q}} = 0, \qquad \vec{q} = \vec{q}_0$$

if $\vec{q}_0$ is a local minimum of $V(\vec{q})$, namely

$$\boxed{\left.\frac{\partial V}{\partial q_i}\right|_{\vec{q}_0} = 0 \qquad \text{for all } i = 1, \ldots, n}$$

and the **Hessian** matrix at $\vec{q} = \vec{q}_0$

$$\boxed{\mathcal{V}_{ij} \equiv \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{\vec{q}_0}}$$

is **positive definite**, i.e.

$$\vec{\eta} \cdot \mathcal{V} \vec{\eta} > 0$$

for any non-zero $\vec{\eta}$. This means that V increases in all directions away from $\vec{q}_0$, so $\vec{q}_0$ is a local minimum.

- Note: $\mathcal{V}_{ij} = \mathcal{V}_{ji}$ (symmetric)

Theorem: For a symmetric matrix the following two conditions are equivalent:

(1) For all non-zero $\vec{\eta}$ one has $\vec{\eta} \cdot \mathcal{V} \vec{\eta} > 0$.

(2) All eigenvalues of $\mathcal{V}$ are strictly positive.

1.6 Lagrangian of small oscillations

$$L = \frac{1}{2} a_{ij}(\vec{q}) \dot{q}_i \dot{q}_j - V(\vec{q})$$

where $i, j = 1, \dots, n$. Expand this around

$$\dot{\vec{q}} = 0, \qquad \vec{q} = \vec{q}_0, \qquad \left. \frac{\partial V}{\partial q_i} \right|_{\vec{q}_0} = 0$$

in fluctuations $\vec{\eta} = \vec{q} - \vec{q}_0$.

Ignoring irrelevant constants and higher-order terms ($\mathcal{O}(\eta^3)$), the Lagrangian is

$$\boxed{L_2 = \frac{1}{2} \mathcal{T}_{ij} \dot{\eta}_i \dot{\eta}_j - \frac{1}{2} \mathcal{V}_{ij} \eta_i \eta_j}$$

Here

$$\boxed{\mathcal{T}_{ij} \equiv a_{ij}(\vec{q}), \qquad \mathcal{V}_{ij} \equiv \left. \frac{\partial^2 V}{\partial q_i \partial q_j} \right|_{\vec{q}_0}}$$

- Both are constant symmetric matrices.
- $\mathcal{V}_{ij}$ is positive definite by assumption.
- For the kinetic energy to be always positive, $\mathcal{T}_{ij}$ is also positive definite.

Euler-Lagrange equations:

$$\frac{\partial L}{\partial \dot{\eta}_l} = \frac{1}{2} \mathcal{T}_{lj} \dot{\eta}_j + \frac{1}{2} \mathcal{T}_{il} \dot{\eta}_i = \mathcal{T}_{lj} \dot{\eta}_j$$
$$\frac{\partial L}{\partial \eta_l} = -\frac{1}{2} \mathcal{V}_{lj} \dot{\eta}_j - \frac{1}{2} \mathcal{V}_{il} \dot{\eta}_i = -\mathcal{V}_{lj} \dot{\eta}_j$$

So we have

$$\boxed{\mathcal{T}_{lj} \ddot{\eta}_j + \mathcal{V}_{lj} \eta_j = 0}$$

for $l = 1, \dots, n$.

1.7 Finding normal modes

Now let us find solutions of

$$\mathcal{T}_{lj} \ddot{\eta}_j + \mathcal{V}_{lj} \eta_j = 0$$

of the form

$$\eta_j = Re[A_j e^{i\omega t}] \qquad A_j \in \mathbb{C}$$

with the *same* ω for all $j = 1, \dots, n$.

As we have discussed, with the understanding that we take the real part at the very end, let us plug this η_j into the Euler-Lagrange equations. We get

$$\boxed{\left(\omega^2 \mathcal{T}_{lj} - \mathcal{V}_{lj} \right) A_j = 0}$$

These are n homogeneous equations in n unknowns A_j. The only solution is $A_j = 0$ for all j, **unless**:

$$\boxed{\det(\omega^2 \mathcal{T} - \mathcal{V}) = 0}$$

This is called the **secular equation**, or the **characteristic equation**. It is an equation for ω.

- It is an algebraic equation of degree n in ω^2. Thus, there are n solutions.
- The solutions (ω) are called **normal frequencies**.
- All $\omega^2 > 0$.
- If we plug each value of ω^2 back into

$$\left(\omega^2 \mathcal{T}_{lj} - \mathcal{V}_{lj}\right) A_j = 0$$

we can find the amplitude A_j for each degree of freedom.

1.8 Procedure to get normal modes

(1) Find an equilibrium position by solving

$$\left.\frac{\partial V}{\partial q_i}\right|_{\vec{q}_0} = 0$$

(2) Compute the matrices

$$\mathcal{T}_{ij} \equiv a_{ij}(\vec{q}), \qquad \mathcal{V}_{ij} \equiv \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{\vec{q}_0}$$

For stability, $\mathcal{V}$ must be positive definite.

(3) Solve the secular equation

$$\det(\omega^2 \mathcal{T} - \mathcal{V}) = 0$$

to determine n normal frequencies ω.

(4) Plug back ω^2 into

$$\left(\omega^2 \mathcal{T}_{lj} - \mathcal{V}_{lj}\right) A_j = 0$$

to get the amplitude $\vec{A}$.

We have to solve $n-1$ out of these n equations to find $\vec{A}$.