SPA5304 Physical Dynamics Lecture 20-21

David Vegh
(figures by Masaki Shigemori)

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1 Spinning tops

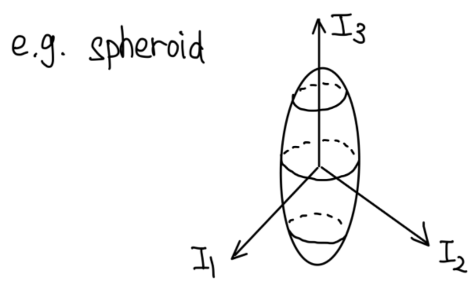
1.1 Free rotation

Set $K_a = 0$. The Euler equations simplify to

$$\begin{aligned}\dot{\omega}_1 + \frac{I_3 - I_2}{I_1}\omega_2\omega_3 &= 0\\ \dot{\omega}_2 + \frac{I_1 - I_3}{I_2}\omega_1\omega_3 &= 0\\ \dot{\omega}_3 + \frac{I_2 - I_1}{I_3}\omega_1\omega_2 &= 0\end{aligned}$$

Let us study these in special cases.

1.1.1 Symmetric top: $I_1 = I_2$



$\dot{\omega}_3 = 0$ implies $\omega_3 =$const.

The other two equations become

$$\begin{cases}\dot{\omega}_1 = -\Omega\omega_2\\ \dot{\omega}_2 = +\Omega\omega_1\end{cases}$$

where $\Omega \equiv \omega_3 \frac{I_3 - I_1}{I_1}$.

Now this can be written as a complex valued equation

$$\frac{d}{dt}(\omega_1 + i\omega_2) = i\Omega(\omega_1 + i\omega_2)$$

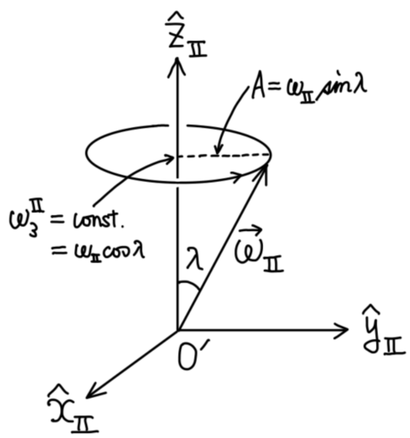
with solution

$$\omega_1 + i\omega_2 = Ae^{i\Omega t}$$

Choosing real A the result is

$$\begin{cases} \omega_1 = A\cos\Omega t \\ \omega_2 = A\sin\Omega t \\ \omega_3 = \text{const.} \end{cases}$$

This motion is called precession.



The $\vec{\omega}_{II}$ vector goes once around in time $T = \frac{2\pi}{\Omega}$.

Note that this precession is with respect to the body axes which are moving themselves. The precession in an inertial frame is different.

1.1.2 Angular momentum

$$\vec{L}_{COM,II} = I\vec{\omega}_{II}, \qquad I_1 = I_2$$

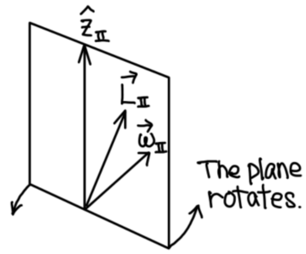
$$\begin{pmatrix} L_1^{II} \\ L_2^{II} \\ L_3^{II} \end{pmatrix} = \begin{pmatrix} I_1\omega_1^{II} \\ I_1\omega_2^{II} \\ I_3\omega_3^{II} \end{pmatrix} = \begin{pmatrix} I_1 A\cos\Omega t \\ I_1 A\sin\Omega t \\ I_3\omega_3^{II} \end{pmatrix}$$

- Similar behavior as $\vec{\omega}_{II}$.
- $\vec{L}$ is conserved because the torque $\vec{K} = 0$. Thus, its components in S_I are constant. The above motion is because S_{II} is moving.
- Note that

$$\vec{\omega} = \underbrace{\omega_1\hat{x} + \omega_2\hat{y}}_{\equiv\vec{\omega}_\perp} + \omega_3\hat{z}$$

$$\vec{L} = I_1\underbrace{(\omega_1\hat{x} + \omega_2\hat{y})}_{\vec{\omega}_\perp} + I_3\omega_3\hat{z}$$

which means that $\vec{L}$, $\vec{\omega}$, and $\vec{z}$ are coplanar since they all lie on the rotating plane spanned by $\hat{z}$ and $\vec{\omega}_\perp$ as shown in the following figure:



• For Earth, Euler's prediction (1765) was

$$\frac{I_3 - I_1}{I_1} \approx \frac{1}{305}$$
$$\omega_3^{II} = \frac{2\pi}{1\text{ day}}$$

which gives

$$\Omega = \omega_3^{II} \frac{I_3 - I_1}{I_1} = \frac{2\pi}{305\text{ days}}$$

and a period of $T = \frac{2\pi}{\Omega} = 305$ days.

But the actual period is 433 days (Chandler's wobble, 1891), due to Earth's non-rigidity.

1.2 Euler equations for an asymmetric top $I_1 \neq I_2 \neq I_3 \neq I_1$

• This is an example of solvable systems, for which the equations of motion can be integrated and the solution can be found explicitly.

• The explicit solution involves elliptic functions (see Landau-Lifshitz §37)

• We will limit ourselves to describe two things:

(i) Qualitative analysis

(ii) Rotation near one of the principal axes

1.2.1 Integrals (constants) of motion

$$\left.\begin{array}{l} \vec{L} : 3 \text{ inertial components} \\ E : 1 \end{array}\right\} \text{4 integrals (constants) of motion}$$

However, we are describing motion in the non-inertial frame S_{II}. The components of $\vec{L}$ in S_{II} are not separately conserved.

An idea: Consider $(\vec{L})^2$ which is the same in S_I and S_{II} and thus is a constant of motion.

Let's denote $L \equiv |\vec{L}|$. Recall that $\vec{L} = I\vec{\omega}$. We will drop the label "II" and denote the components of $\vec{\omega}$ in S_{II} by $(\omega_1, \omega_2, \omega_3)$. Then,

$$L^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$$

Also,

$$E \equiv T = \frac{1}{2}\vec{\omega} \cdot I\vec{\omega} = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$$

We thus get two integrals of motion:

$$\boxed{L^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2}$$

$$\boxed{2E = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2}$$

Or, experessing $\vec{\omega}$ in terms of $\vec{L}$,

$$\begin{aligned} L^2 &= L_1^2 + L_2^2 + L_3^2 \\ 2E &= \frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3} \end{aligned}$$

where $(L_1,\ L_2,\ L_3)$ are the components of $\vec{L}$ in S_{II}.

1.2.2 The solution (in principle)

Given the L^2 and E constants, we could solve the equations for, say, ω_1 and ω_2:

$$\begin{cases} \omega_1 = \omega_1(\omega_3;\ L^2,\ E) \\ \omega_2 = \omega_2(\omega_3;\ L^2,\ E) \end{cases}$$

Then one could take

$$I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 = f(\omega_3)$$

and find ω_3 as a solution to this differential equation.

This lease to a complete solution (see Landau-Lifshitz §37)

1.3 Rotation near one of the principal axes

The general rotation is complicated. Instead, study the following stability problem.

- Consider Euler's equations for a rigid body with generic $I_{1,2,3}$.

$$\begin{aligned} \dot{\omega}_1 + \frac{I_3 - I_2}{I_1}\omega_2\omega_3 &= 0 \\ \dot{\omega}_2 + \frac{I_1 - I_3}{I_2}\omega_1\omega_3 &= 0 \\ \dot{\omega}_3 + \frac{I_2 - I_1}{I_3}\omega_1\omega_2 &= 0 \end{aligned}$$

The following are solutions:

$$\omega_1 = \text{const.}, \qquad \omega_2 = \omega_3 = 0$$

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$$\omega_3 = \text{const.}, \qquad \omega_1 = \omega_2 = 0$$

Namely, rotation around one of the principal axes is in fact a solution of Euler's equations.

• Question: Are these solutions stable? If we consider small perturbations around such a solution, will the system stay close to the original solution?

• Assume: At the initial time, the rigid body rotates about an axis *near* one of its principal axes, say $\hat{x}_3$. Then,

$$\begin{aligned} \omega_1 &= \epsilon_1(t) \\ \omega_2 &= \epsilon_2(t) \\ \omega_3 &= \omega_0 + \epsilon_3(t) \end{aligned}$$

where at $t \approx 0$: $\epsilon_{1,2,3} \ll \omega_0$.

We want to now the motion for small times after $t = 0$. Plugging the above ω_i into Euler's equations,

$$I_1 \dot{\omega}_1 = I_1 \dot{\epsilon}_1 = (I_2 - I_3)\omega_2\omega_3 \approx (I_2 - I_3)\omega_0\epsilon_2$$

$$I_2 \dot{\omega}_2 = I_2 \dot{\epsilon}_2 = (I_3 - I_1)\omega_1\omega_3 \approx (I_3 - I_1)\omega_0\epsilon_1$$

$$I_3 \dot{\omega}_3 = I_3 \dot{\epsilon}_3 = (I_1 - I_2)\omega_1\omega_2 \approx 0$$

Here we have neglected terms of order $\mathcal{O}(\epsilon^2)$. The equations are

$$\dot{\epsilon}_1 = \frac{I_2 - I_3}{I_1}\omega_0\epsilon_2$$

$$\dot{\epsilon}_2 = \frac{I_3 - I_1}{I_2}\omega_0\epsilon_1$$

$$\dot{\epsilon}_3 = 0$$

Take one more time-derivative:

$$\ddot{\epsilon}_1 = \frac{I_2 - I_3}{I_1}\omega_0\dot{\epsilon}_2 = \frac{(I_2 - I_3)(I_3 - I_1)}{I_1 I_2}\omega_0^2\epsilon_1$$

$$\ddot{\epsilon}_2 = \frac{I_3 - I_1}{I_2}\omega_0\dot{\epsilon}_1 = \frac{(I_3 - I_1)(I_2 - I_3)}{I_1 I_2}\omega_0^2\epsilon_2$$

so we get

$$\ddot{\epsilon}_1 = -\Omega^2\epsilon_1$$

$$\ddot{\epsilon}_2 = -\Omega^2\epsilon_2$$

$$\dot{\epsilon}_3 = 0$$

where

$$\Omega^2 \equiv \omega_0^2\frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}$$

which is not necessarily positive!

Two cases:

(i) $\Omega^2 > 0 \quad \Leftrightarrow \quad (I_3 - I_2)(I_3 - I_1) > 0$

which means $I_3 < I_1, I_2$ or $I_3 > I_1, I_2$.

Then, the motion is oscillatory: $\epsilon_{1,2} \propto e^{i\Omega t}$

If the amplitude is small at $t = 0$, then it will remain small.

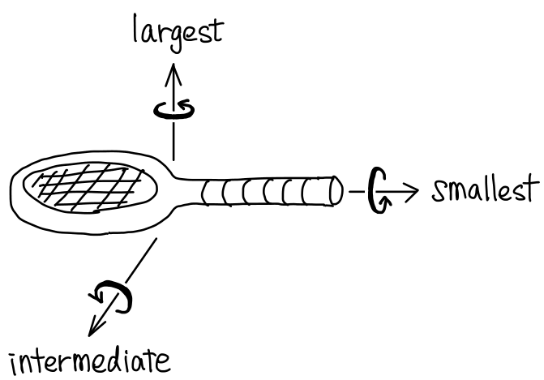
> The motion about an axis near the one with the **smallest** or **largest** principal moment is stable.

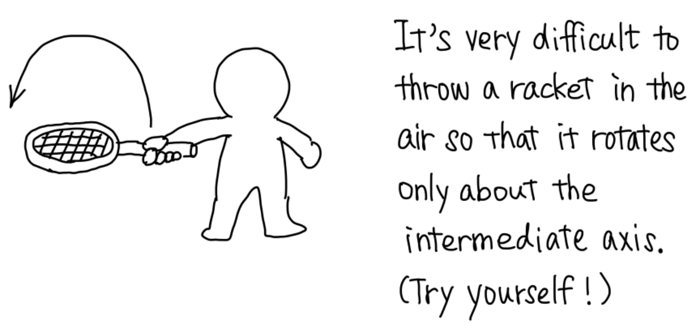
(ii) $\Omega^2 < 0 \quad \Leftrightarrow \quad I_1 < I_3 < I_2$ or $I_2 < I_3 < I_1$

In this case we see exponential growth: $\epsilon_{1,2} \propto e^{+\sqrt{|\Omega^2|}t}$

> The motion about an axis near the **intermediate** axis is unstable.

- As $\epsilon_{1,2}$ grow, the condition that they are small is violated and one should resort to the exact solution.
- The above result is called the "intermediate axis theorem", or the "'tennis racket theorem"





For further discussions of rigid bodies, see the lecture notes from 2016/17.

2 Small Oscillations

• Physical systems display equilibrium configurations.

• Equilibrium means that the system remains in that position/configuration indefinitely, when placed there with no initial velocities.

• When the equilibrium is stable, a small perturbation (i.e. a small kick) results in **oscillations about the equilibrium position**.

• To first order approximation, the oscillations are **harmonic**: single constant-frequency oscillations independent of amplitude.

• Consider a Lagrangian system described by certain generalized coordinates:

$$\vec{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \qquad \qquad (n \text{ degrees of freedom})$$

$$\begin{cases} \vec{r}_1 = \vec{r}_1(q_1, \ldots, q_n) \\ \vdots \\ \vec{r}_N = \vec{r}_N(q_1, \ldots, q_n) \end{cases}$$

2.1 Kinetic energy

To discuss oscillations in general systems, let us find the general form of the kinetic energy in terms of $\vec{q}$.

$$T = \frac{1}{2} \sum_{k=1}^{N} m_k \dot{\vec{r}}_k^2$$

$$\dot{\vec{r}}_k = \frac{d}{dt} \vec{r}_k(q_1, \ldots, q_n) = \sum_i \frac{\partial \vec{r}_k}{\partial q_i} \dot{q}_i$$

Thus,

$$T = \frac{1}{2} \sum_{k=1}^{N} \sum_{i,j=1}^{n} m_k \frac{\partial \vec{r}_k}{\partial q_i} \cdot \frac{\partial \vec{r}_k}{\partial q_j} \dot{q}_i \dot{q}_j \equiv \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(\vec{q}) \dot{q}_i \dot{q}_j$$

So

$$\boxed{T = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(\vec{q}) \dot{q}_i \dot{q}_j}$$

where

$$\boxed{a_{ij}(\vec{q}) \equiv \sum_{k=1}^{N} m_k \frac{\partial \vec{r}_k}{\partial q_i} \cdot \frac{\partial \vec{r}_k}{\partial q_j}}$$

Note that a is symmetric: $a_{ij} = a_{ji}$.

For the potential we will always assume $V = V(\vec{q})$.

- Theorem: The configuration $\dot{\vec{q}} = 0$, $\vec{q} = \vec{q}_0$ is a solution of the Euler-Lagrange equations if

$$\left.\frac{\partial V}{\partial \vec{q}}\right|_{\vec{q}=\vec{q}_0} = 0$$

This is an equilibrium position.

Proof:

The Euler-Lagrange equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{q}}} - \frac{\partial L}{\partial \vec{q}} = 0$$

Here because V does not contain $\dot{\vec{q}}$ (which was our assumption),

$$\frac{\partial L}{\partial \dot{\vec{q}}} = \frac{\partial T}{\partial \dot{\vec{q}}}$$

Now at $\dot{\vec{q}} = 0$, $\vec{q} = \vec{q}_0$

$$T = \frac{1}{2} a_{ij}(\vec{q})\dot{q}_i\dot{q}_j = 0$$

$$\frac{\partial T}{\partial \dot{q}_i} = a_{ij}\dot{q}_j = 0$$

$$\frac{\partial T}{\partial q_i} = \frac{\partial a_{jk}}{\partial q_i}\dot{q}_j\dot{q}_k = 0$$

Therefore,

$$\left.\frac{\partial L}{\partial \vec{q}}\right|_{\vec{q}=\vec{q}_0} = \left.\left(\frac{\partial T}{\partial \vec{q}} - \frac{\partial V}{\partial \vec{q}}\right)\right|_{\vec{q}=\vec{q}_0} = 0 - \left.\frac{\partial V}{\partial \vec{q}}\right|_{\vec{q}=\vec{q}_0} = 0 \qquad \text{(by assumption)}$$

Since we have seen that $\frac{\partial L}{\partial \dot{\vec{q}}} = 0$, we have that

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{q}}} - \frac{\partial L}{\partial \vec{q}} = 0$$

Q.E.D.

2.2 Examples of $a_{ij}(\vec{q})$

2.2.1 Spherical polar coordinates

$$\vec{r} = (x, y, z) \to (r, \theta, \phi)$$

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

- We could compute a_{ij} using the definition

$$a_{ij}(\vec{q}) \equiv \sum_{k=1}^{N} m_k \frac{\partial \vec{r}_k}{\partial q_i} \cdot \frac{\partial \vec{r}_k}{\partial q_j}$$

but in concrete cases like this, it is easy to compute T:

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{m}{2}(\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)$$

and read off a_{ij} from it.

- Reading off a_{ij}:

$$T = \frac{1}{2}(\dot{r} \;\; \dot{\theta} \;\; \dot{\phi}) \begin{pmatrix} m & 0 & 0 \\ 0 & mr^2 & 0 \\ 0 & 0 & mr^2\sin^2\theta \end{pmatrix} \begin{pmatrix} \dot{r} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

From this

$$a_{ij}(r,\ \theta,\ \phi) = \begin{pmatrix} m & 0 & 0 \\ 0 & mr^2 & 0 \\ 0 & 0 & mr^2\sin^2\theta \end{pmatrix}$$

2.2.2 With a constraint

If we have the constraint $r = \text{fixed} \equiv l$, then the generalized coordinates can be $(\theta,\ \phi)$.

$$T = \frac{ml^2}{2}(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) = \frac{1}{2}(\dot{\theta} \;\; \dot{\phi}) \begin{pmatrix} ml^2 & 0 \\ 0 & ml^2\sin^2\theta \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

and thus

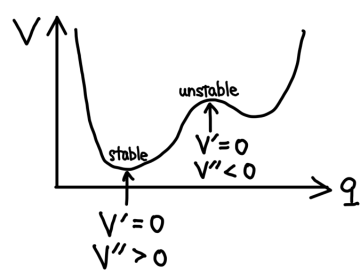
$$a_{ij} = \text{diag}(ml^2,\ ml^2\sin^2\theta)$$

2.3 One degree of freedom

$$L = T - V = \frac{1}{2}a(q)\dot{q}^2 - V(q)$$

$$\text{Equilibrium:} \quad q = q_0, \quad \dot{q} = 0, \quad V'(q_0) = 0$$

The equilibrium position is stable if $q = q_0$ is the minimum of the potential, i.e. $V''(q_0) > 0$ (generically).



Let us look at this in more detail. Consider a small fluctuation around $q = q_0$ and Taylor-expand the Lagrangian in

$$\eta \equiv q - q_0$$

Since T is quadratic, we only need to expand the potential up to 2nd order in η.

$$V(q) = V(q_0) + \eta \underbrace{V'(q_0)}_{\text{vanishes by assumption}} + \frac{1}{2}\eta^2 V''(q_0) + \ldots$$

$V(q_0)$ is an irrelevant constant in the potential. The Lagrangian is

$$L = \frac{1}{2}a(q_0)\dot{\eta}^2 - \frac{1}{2}\eta^2 V''(q_0) + \text{const.} + \mathcal{O}(\eta^3)$$

Note: we approximated $a(q) \approx a(q_0)$ at this order.

If we set

$$a(q_0) \equiv m > 0$$

$$V''(q_0) \equiv m\omega^2 > 0$$

then the 2nd order Lagrangian is

$$\boxed{L = \frac{1}{2}m\dot{\eta}^2 - \frac{1}{2}m\omega^2\eta^2}$$

which is the same as the Lagrangian of a simple harmonic oscillator.

The Euler-Lagrange equation is

$$\boxed{\ddot{\eta} + \omega^2\eta = 0}$$

and

$$\omega = \sqrt{\frac{V''(q_0)}{a(q_0)}}$$

is the frequency.

The general solution is

$$\eta(t) = c_1 \cos\omega t + c_2 \sin\omega t$$

or equivalently:

$$\eta(t) = a\cos(\omega t + \alpha)$$

where a is the **amplitude** and α is the **phase**:

$$a = \sqrt{c_1^2 + c_2^2}\ , \qquad \tan\alpha = -c_2/c_1.$$

We can also write this as

$$\eta(t) = Re[Ae^{i\omega t}], \qquad A = ae^{i\alpha} \in \mathbb{C}$$

A is the complex amplitude.

Since the complex function $\eta(t) = Ae^{i\omega t}$ is also a solution to the Euler-Lagrange equation, we can look for a complex solution of the form $\eta \propto e^{i\omega t}$ and take the real part at the end of the computation to get a physical solution. In practice, this is very useful, because the action of derivatives are much simpler on $e^{i\omega t}$ than on sine and cosine functions.

2.4 Summary

- $\dot{q} = 0$, $q = q_0$ is a stable equilibrium position iff $V'(q_0) = 0$ and $V''(q_0) > 0$.
- In this case, if we write $q = q_0 + \eta$ where η is small, then η will do small harmonic oscillations.
- The frequency of oscillations is

$$\omega = \sqrt{\frac{V''(q_0)}{a(q_0)}}$$

- The period is $T = \frac{2\pi}{\omega}$.
- If $V'(q_0) = 0$ but $V''(q_0) < 0$, then $q = q_0$ is an unstable equilibrium position. In this case

$$\omega^2 \equiv \frac{V''(q_0)}{a(q_0)} < 0$$

The Euler-Lagrange equation is

$$\ddot{\eta} - |\omega^2|\eta = 0$$

which has solutions

$$\eta = Ae^{-|\omega|t} + Be^{+|\omega|t}$$

This shows a runaway behavior (exponential growth) $\Rightarrow$ instability.