SPA5304 Physical Dynamics Lecture 19

David Vegh
(figures by Masaki Shigemori)

26 February 2019

1 Rigid bodies

1.1 Parallel axis theorem

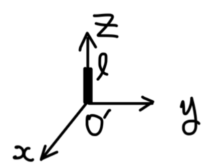
If the axis $\hat{n}_2$ goes through the center of mass, then the moment of inertia about another parallel axes $\hat{n}_1$ is given by

$$I_{\hat{n}_1} = I_{\hat{n}_2} + Ma^2$$

where M is the total mass of the rigid body and a is the distance between the two axes.

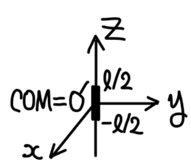
1.1.1 An example: the rod

- If O' is at the end of the rod:



$$I_{O'} = \begin{pmatrix} \frac{Ml^3}{3} & 0 & 0 \\ 0 & \frac{Ml^3}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- If O' is at the center-of-mass:



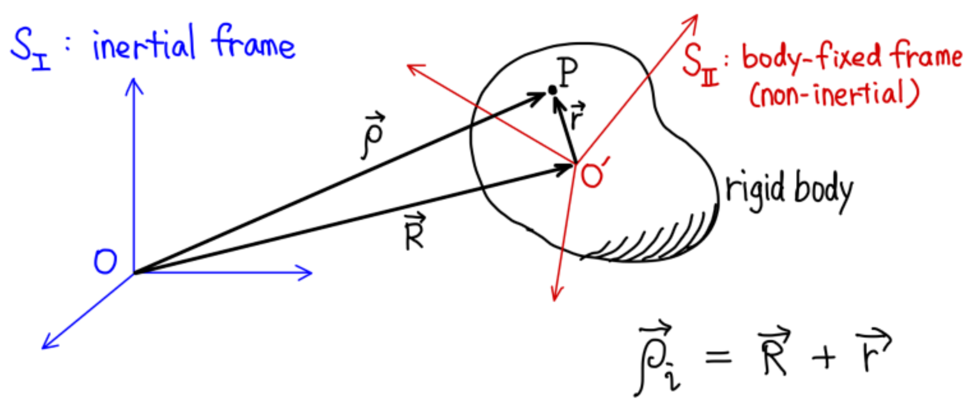
$$I_{COM} = \begin{pmatrix} \frac{Ml^3}{12} & 0 & 0 \\ 0 & \frac{Ml^3}{12} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We have shifted the axis by $a = l/2$:

$$
\begin{aligned}
I_{xx}^{O'} &= I_{xx}^{COM} + m\left(\frac{l}{2}\right)^2 \\
I_{yy}^{O'} &= I_{yy}^{COM} + m\left(\frac{l}{2}\right)^2 \\
I_{zz}^{O'} &= I_{zz}^{COM}
\end{aligned}
$$

Rotation about $\hat{z}$ is not affected.

1.2 Angular momentum



The fundamental formula of rigid kinematics:

$$\dot{\vec{\rho}}_i = \dot{\vec{R}} + \vec{\omega} \times \vec{r}_i$$

The angular momentum about O is

$$\vec{L}_O = \sum_i \vec{\rho}_i \times m_i \dot{\vec{\rho}}_i = \sum_i (\vec{R} + \vec{r}_i) \times m_i (\dot{\vec{R}} + \vec{\omega} \times \vec{r}_i)$$

$$= \vec{R} \times \sum_i m_i \dot{\vec{R}} + \vec{R} \times (\vec{\omega} \times \sum_i m_i \vec{r}_i) + (\sum_i m_i \vec{r}_i) \times \dot{\vec{R}} + \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$$

Let us take $O' = COM$. Then $\sum_i m_i \vec{r}_i = 0$ and we get

$$\vec{L}_O = \vec{R} \times M\dot{\vec{R}} + \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$$

Now using the formula

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

with $\vec{A} = \vec{C} = \vec{r}_i$ and $\vec{B} = \vec{\omega}$,

$$\vec{L}_O = \vec{R} \times M\dot{\vec{R}} + \sum_i m_i \left(\vec{r}_i^2 \vec{\omega} - (\vec{r}_i \cdot \vec{\omega})\vec{r}_i\right)$$

The second term can be written as

$$\underbrace{\sum_i m_i \left(\vec{r}_i^2 \delta_{ab} - r_{ia} r_{ib}\right) \omega_b}_{I_{COM}}$$

Here a, b are vector indices. Thus,

$$\boxed{\vec{L}_O = \vec{R} \times M\dot{\vec{R}} + \underbrace{I_{COM}\vec{\omega}}_{\vec{L}_{COM}}}$$

Compare this with our earlier formula for the angular momentum of a system of particles:

$$\vec{L} = \vec{R} \times M\dot{\vec{R}} + \vec{L}'$$

We see that L_{COM} is nothing but $\vec{L}'$, i.e. the angular momentum with respect to the COM ("spin part")

1.3 Kinetic energy in terms of $\vec{L}$

- Recall that for $O' = COM$,

$$T = \frac{1}{2} M\dot{\vec{R}}^2 + \frac{1}{2}\omega \cdot \underbrace{I_{COM}\vec{\omega}}_{\vec{L}_{COM}}$$

Thus

$$\boxed{T = \frac{1}{2} M\dot{\vec{R}}^2 + \frac{1}{2}\omega \cdot \vec{L}_{COM}}$$

- For a fixed O' we have

$$T = \frac{1}{2}\vec{\omega} \cdot I_{O'}\vec{\omega}$$

Since O' is fixed, a point P_i in the rigid body has velocity

$$\dot{\vec{r}}_i = \vec{\omega} \times \vec{r}_i$$

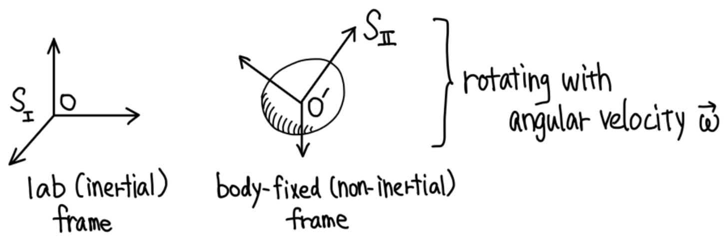
Therefore,

$$\vec{L}_{O'} = \sum_i \vec{r}_i \times m_i(\vec{\omega} \times \vec{r}_i) = \sum_i m_i \left(\vec{r}_i^2 \vec{\omega} - (\vec{r}_i \cdot \vec{\omega})\vec{r}_i\right) = I_{O'}\vec{\omega}$$

So we have

$$\boxed{T = \frac{1}{2}\vec{\omega} \cdot \vec{L}_{O'}}$$

2 Spinning Tops



Let us study a more general (3-dimensional) motion of rigid bodies. We immediately face a problem: $I_{ab}^{O'}$ is simple in the body-fixed (non-inertial) frame S_{II}, but our formulation has been about an inertial frame (e.g. S_I). We need to find awa to describe dynamics in the body-fixed frame.

2.1 Rotating frame

Consider some general vector $\vec{u}$. If it is not changing in the moving frame S_{II}, its rate of change is due only to the rotation of the frame

$$\frac{d\vec{u}}{dt} = \vec{\omega} \times \vec{u}$$

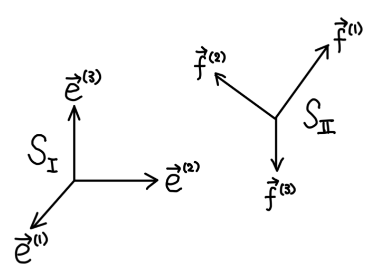
More generally, if $\vec{u}$ is changing in the moving frame,

$$\frac{d\vec{u}}{dt} = \underbrace{\frac{d'\vec{u}}{dt}}_{\substack{\text{change w.r.t.} \\ \text{the moving frame } S_{II}}} + \vec{\omega} \times \vec{u}$$

Let us make this derivation more precise.

Take bases for frames S_I and S_{II}:

$$\begin{array}{ccccc} S_I & : & \vec{e}^{(a)} & a = 1,2,3 & \text{(fixed)} \\ S_{II} & : & \vec{f}^{(a)} & a = 1,2,3 & \text{(moving)} \end{array}$$



A general vector $\vec{u}$ can be expanded as

$$\boxed{\vec{u} = u_a^I \vec{e}^{(a)} = u_a^{II} \vec{f}^{(a)}}$$

where u_a^I and u_a^{II} are the components in S_I and S_{II}, respectively.

The time-derivative is

$$\dot{\vec{u}} = \dot{u}_a^I \vec{e}^{(a)} = \dot{u}_a^{II} \vec{f}^{(a)} + u_a^{II} \dot{\vec{f}}^{(a)}$$

Since S_{II} is rotating with angular velocity $\vec{\omega}$,

$$\dot{\vec{f}}^{(a)} = \vec{\omega} \times \vec{f}^{(a)}$$

and thus

$$\dot{\vec{u}} = \frac{d\vec{u}}{dt} = \underbrace{\dot{u}_a^{II} \vec{f}^{(a)}}_{\frac{d'\vec{u}}{dt}} + \vec{\omega} \times \underbrace{u_a^{II} \vec{f}^{(a)}}_{\vec{u}}$$

If we define

$$\vec{u}_{II} = \begin{pmatrix} u_1^{II} \\ u_2^{II} \\ u_3^{II} \end{pmatrix} \quad : \qquad \text{components in } S_{II}$$

Then

$$\boxed{\dot{\vec{u}} = \underbrace{\dot{\vec{u}}_{II}}_{\frac{d}{dt} \text{ of components in } S_{II}} + \vec{\omega}_{II} \times \vec{u}_{II}}$$

We succeeded in expressing dynamics in terms of quantities in the moving frame S_{II}.

2.2 Euler equations

Recall:

$$\vec{L} = \sum_i \vec{r}_i \times m_i \dot{\vec{r}}_i$$

$$\dot{\vec{L}} = \sum_i \vec{r}_i \times \vec{F}_i^{(e)}$$

In particular, for angular momentum about the COM,

$$\dot{\vec{L}}_{COM} = (\text{torque about COM}) \equiv \vec{K}$$

From the above formula,

$$\dot{\vec{L}}_{COM,II} + \vec{\omega}_{II} \times \vec{L}_{COM,II} = \vec{K}_{II}$$

We know that $\vec{L}_{COM} = I_{COM}\vec{\omega}$.

If we take S_{II} to be a principal axis system, then we get

$$\vec{L}_{COM,II} = \begin{pmatrix} I_1\omega_1^{II} \\ I_2\omega_2^{II} \\ I_3\omega_3^{II} \end{pmatrix}$$

Furthermore,

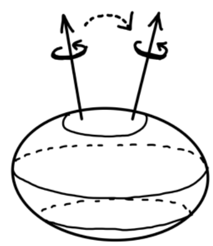
$$\vec{\omega}_{II} \times \vec{L}_{COM,II} = \begin{pmatrix} \omega_1^{II} \\ \omega_2^{II} \\ \omega_3^{II} \end{pmatrix} \times \begin{pmatrix} I_1\omega_1^{II} \\ I_2\omega_2^{II} \\ I_3\omega_3^{II} \end{pmatrix} = \begin{pmatrix} (I_3 - I_2)\omega_2^{II}\omega_3^{II} \\ (I_1 - I_3)\omega_1^{II}\omega_3^{II} \\ (I_2 - I_1)\omega_1^{II}\omega_2^{II} \end{pmatrix}$$

Thus we have obtained the **Euler equations**

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= K_1 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 &= K_2 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= K_3 \end{aligned}$$

Here the index II has been suppressed on ω and K.

- These are non-linear differential equations.
- They describe the motion of $\vec{\omega}$ in the body-fixed frame with $O' = COM$.



The axis about which the rigid body rotates keeps changing in the body-fixed frame.