SPA5304 Physical Dynamics Lecture 16

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1 Symmetries and conservation laws

1.1 Energy conservation

Assuming the Lagrangian does not depend explicitly on time, we have seen that

$$\boxed{H \equiv \vec{p}\cdot\dot{\vec{q}} - L}$$

is conserved. H is called the **Hamiltonian**.

- If L depends explicitly on t, then we ehave to be more careful with the derivation:

$$\delta L = L(\vec{q}+\delta\vec{q},\dot{\vec{q}}+\delta\dot{\vec{q}},t+\delta t) - L(\vec{q},\dot{\vec{q}},t) = \frac{\partial L}{\partial\vec{q}}\cdot\delta\vec{q} + \frac{\partial L}{\partial\dot{\vec{q}}}\cdot\delta\dot{\vec{q}} + \underbrace{\frac{\partial L}{\partial t}\delta t}_{\text{extra term}} = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\vec{q}}}\cdot\delta\vec{q}\right) + \frac{\partial L}{\partial t}\delta t = \frac{d}{dt}\delta F$$

Thus we get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\vec{q}}}\cdot\delta\vec{q} - \delta F\right) = -\frac{\partial L}{\partial t}\delta t \neq 0$$

or

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

This means that changing parameters of the system injects/extracts energy.

1.2 The meaning of H

H is nothing but the energy is many cases. Let's see some examples

(i) If $\vec{q}$ are Cartesian coordinates,

$$T = \frac{1}{2}m\dot{\vec{r}}^2 \qquad \text{and} \qquad \vec{p} = m\dot{\vec{r}}$$

Then

$$H = \vec{p}\cdot\dot{\vec{q}} - L = m\dot{\vec{r}}\cdot\dot{\vec{r}} - \left(\frac{1}{2}m\dot{\vec{r}}^2 - V\right) = \frac{1}{2}m\dot{\vec{r}}^2 + V = E$$

(ii) If $\vec{q}$ are generalized coordinates and T is quadratic in $\dot{q}_i$

$$T = \sum_{i,j} \frac{1}{2} a_{ij}(\vec{q}) \dot{q}_i \dot{q}_j$$

Euler's theorem on homogeneous functions says[1]

$$\sum_i \frac{\partial T}{\partial \dot{q}_i} \dot{q}_i = 2T$$

Hence,

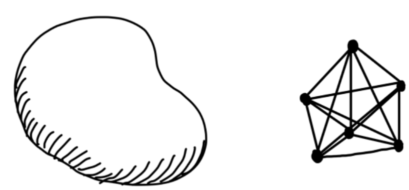
$$H = \vec{p} \cdot \dot{\vec{q}} - L = \frac{\partial L}{\partial \dot{\vec{q}}} \cdot \dot{\vec{q}} - L = 2T - (T - V) = T + V = E$$

2 Rigid bodies

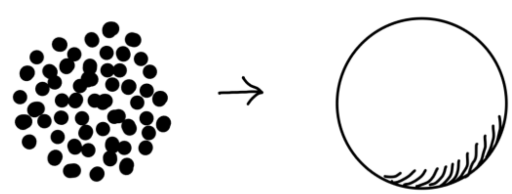
A rigid body is a mechanical model for solid bodies with finite size[2].

- It is represented by a system of particles such that the distances between the particles do not vary.

$$|\vec{r}_i - \vec{r}_j| = C_{ij} = \text{const.}$$



- We will often take the a continuum limit in which the number of particles is infinite.



- We want to study rigid bodies using Lagrangian mechanics. This involves:

(i) Finding the #DoFs, and identifying the generalized coordinates

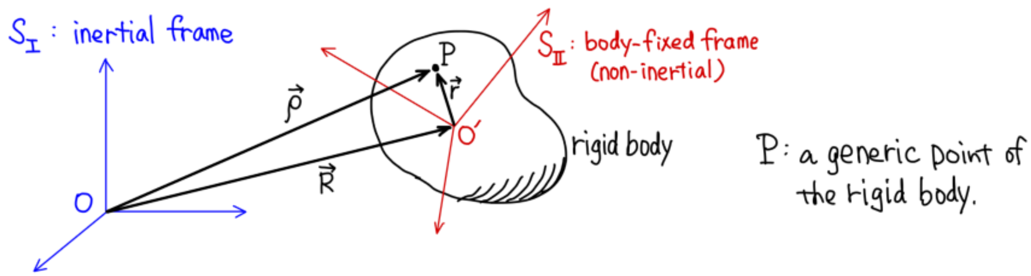
(ii) Finding the kinetic energy T

(iii) Constructing the Lagrangian $L = T - V$

[1] $f(x)$ is a homogeneous function of x of degree s if $f(ax) = a^s f(x)$. In this case, $x\frac{\partial f}{\partial x} = sf$.
[2] "Finite" usually means non-zero (not infinitesimal) and not infinitely large either: somewhere in between.

2.1 Body-fixed frame

To describe the motion of a rigid body, let us introduce a coordinate frame that is fixed to the rigid body and moves with it.



- S_I is an inertial frame (laboratory frame)
- S_{II} is the coordinate system fixed to the rigid body: this is non-inertial.
- The origin O' of S_{II} is not necessarily the COM of the rigid body.

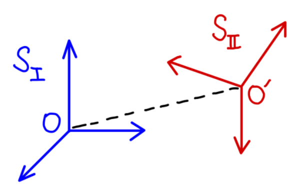
$$\underbrace{\vec{\rho}}_{\substack{\text{position}\\ \text{relative to } O}} = \underbrace{\vec{R}}_{\substack{\text{position}\\ \text{of } O'}} + \underbrace{\vec{r}}_{\substack{\text{position}\\ \text{relative to } O'}}$$

- Studying the motion of a rigid body is equivalent to studying the motion of S_{II} with respect to S_I.

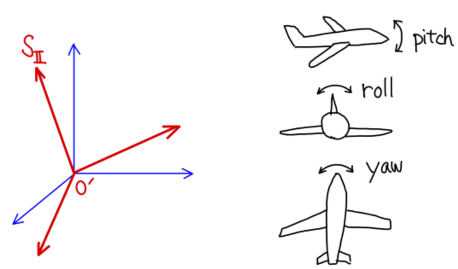
2.2 The number of degrees of freedom

How many parameters do we need to specify the configuration of S_{II} relative to S_I?

(i) The position of O' relative to O is three parameters:



(ii) Rotating the axes of S_{II} relative to those of S_I gives another three parameters (angles)



Therefore we need

$$\boxed{6 \text{ parameters} = 3 \text{ translations} + 3 \text{ rotations}}$$

2.3 Fundamental formula of rigid kinematics

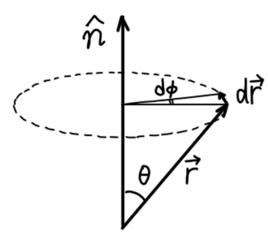
The infinitesimal change of the position P

$$\vec{\rho} = \vec{R} + \vec{r}$$

in an infinitesimal time dt is given by

$$d\vec{\rho} = d\vec{R} + d\vec{r}$$

- $d\vec{R}$ is a translation of O' relative to O.
- $d\vec{r}$ is a rotation by angle $d\phi$ around a certain instantaneous axis $\hat{n}$ passing through O'



and we have

$$d\vec{r} = \hat{n} d\phi \times \vec{r}$$

Plugging this in gives

$$d\vec{\rho} = d\vec{R} + \hat{n} d\phi \times \vec{r}$$

Dividing both sides by dt:

$$\boxed{\dot{\vec{\rho}} = \dot{\vec{R}} + \vec{\omega} \times \vec{r}}$$

This is the fundamental formula of rigid kinematics.

- $\dot{\vec{R}}$ is the velocity of O' relative to O.
- $\vec{\omega}$ is the **angular velocity** of S_{II} about its origin O':

$$\omega \equiv \hat{n} \frac{d\phi}{dt}$$

- We have decomposed the motion of P as a combination of translational and rotational motions.
- The above equation is a vector equation. Components of the equation may be obtained by projecting the vectors on the axes of either S_I or S_{II}.
- The formula is true for all points of the rigid body. For point P_i

$$\dot{\vec{\rho}}_i = \dot{\vec{R}} + \vec{\omega} \times \vec{r}_i$$

and $\vec{\omega}$ is the same for all i. Thus, we can talk about the **angular velocity of the rigid body**.

2.4 Kinetic energy

$$
\begin{aligned}
T &= \frac{1}{2}\sum_i m_i \dot{\vec{\rho}}_i^{\,2} = \frac{1}{2}\sum_i m_i(\dot{\vec{R}} + \vec{\omega}\times\vec{r}_i)^2 \\
&= \frac{1}{2}\sum_i m_i \dot{\vec{R}}^2 + \frac{1}{2}\sum_i m_i(\vec{\omega}\times\vec{r}_i)^2 + \sum_i m_i \dot{\vec{R}}\cdot(\vec{\omega}\times\vec{r}_i) \\
&= \frac{1}{2}\sum_i m_i \dot{\vec{R}}^2 + \frac{1}{2}\sum_i m_i(\vec{\omega}\times\vec{r}_i)^2 + \underbrace{\dot{\vec{R}}\cdot(\vec{\omega}\times\sum_i m_i\vec{r}_i)}_{\text{cross term}}
\end{aligned}
$$

Two situations in which the cross term vanishes:

(A) If O' is fixed, e.g. a physical pendulum with suspension point O'. Then,

$$T = \frac{1}{2}\sum_i m_i(\vec{\omega}\times\vec{r}_i)^2$$

(B) If O' is the center of mass. In this case, $\sum_i m_i\vec{r}_i = 0$.

$$T = \frac{1}{2}\sum_i m_i \dot{\vec{R}}^2 + \frac{1}{2}\sum_i m_i(\vec{\omega}\times\vec{r}_i)^2$$

2.5 Components

So far the expressions for T involved vectors and were valid in any frame.

Let us now introduce components of the vectors. The components depend on the reference frame.

$$\vec{r}_i = \begin{pmatrix} r_{i1} \\ r_{i2} \\ r_{i3} \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}$$

$$\vec{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

$i, j, \ldots$ denote particle number and run over $1 \ldots N$,

$a, b, \ldots$ denote component number and run over $1, 2, 3$ or x, y, z.

At this point we do not specify which frame we are referring to.

T involves $(\vec{\omega} \times \vec{r}_i)^2$. Let us write this quantity in component form. Suppressing the particle number i and using the Einstein summation convention (i.e. repeated indices are summed over)

$$(\vec{\omega} \times \vec{r})^2 = (\vec{\omega} \times \vec{r})_a (\vec{\omega} \times \vec{r})_a = \epsilon_{abc}\omega_b r_c \epsilon_{ade}\omega_d r_e$$

where we have used $(\vec{A} \times \vec{B})_i = \epsilon_{ijk} A_j B_k$.

What is $\epsilon_{abc}\epsilon_{ade}$? Note that

$$\epsilon_{abc}\epsilon_{ade} = \epsilon_{1bc}\epsilon_{1de} + \epsilon_{2bc}\epsilon_{2de} + \epsilon_{3bc}\epsilon_{3de}$$

The first term is non-vanishing iff

$$\{b, c\} = \{d, e\} = \{2, 3\}$$

namely if

(i) $(b, c) = (d, e) = (2, 3)$ this gives $+1$

(ii) $(b, c) = (d, e) = (3, 2)$ this gives $+1$

(iii) $(b, c) = (e, d) = (2, 3)$ this gives -1

(iv) $(b, c) = (e, d) = (3, 2)$ this gives -1

and similarly for the 2nd and 3rd terms. These can be summarized in

$$\epsilon_{abc}\epsilon_{ade} = \delta_{bd}\delta_{ce} - \delta_{be}\delta_{cd}$$

Using this result,

$$(\vec{\omega} \times \vec{r})^2 = \omega_b r_c \omega_b r_c - \omega_b r_c \omega_c r_b = \vec{\omega}^2 \vec{r}^2 - (\vec{\omega} \cdot \vec{r})^2$$

So for the two cases in which the cross term vanished:

(A)

$$T = \frac{1}{2} \sum_i m_i (\vec{\omega}^2 \vec{r}^2 - (\vec{\omega} \cdot \vec{r})^2)$$

This is the rotational energy around the fixed point O'.

(B)

$$T = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \sum_i m_i (\vec{\omega}^2 \vec{r}^2 - (\vec{\omega} \cdot \vec{r})^2)$$

This is the kinetic energy of the COM plus the rotational energy around the COM.

2.6 Inertia tensor

Let us focus on the rotational part of T:

$$\sum_i m_i(\vec{\omega}^2\vec{r}_i^2 - (\vec{\omega}\cdot\vec{r}_i)^2) = \sum_i m_i\omega_a\omega_b\left(\delta_{ab}\vec{r}_i^{\,2} - r_{ia}r_{ib}\right)$$

ω_a has no particle index i. Thus, this can be written as

$$= \underbrace{\left[\sum_i m_i\left(\delta_{ab}\vec{r}_i^{\,2} - r_{ia}r_{ib}\right)\right]}_{I_{ab}}\omega_a\omega_b \equiv I_{ab}\omega_a\omega_b = \vec{\omega}\cdot I\vec{\omega}$$

I is the moment of inertia tensor.

$$\boxed{I_{ab} = \sum_i m_i\left(\delta_{ab}\vec{r}_i^{\,2} - r_{ia}r_{ib}\right) = \sum_i m_i \begin{pmatrix} y_i^2+z_i^2 & -x_iy_i & -x_iz_i \\ -x_iy_i & x_i^2+z_i^2 & -y_iz_i \\ -x_iz_i & -y_iz_i & x_i^2+y_i^2 \end{pmatrix}_{ab}}$$

The inertia tensor is

- symmetric: $I_{ab} = I_{ba}$
- additive: $I_{\{m_i\}} = \sum_i I_{m_i}$

2.6.1 Expressions for the kinetic energy

So again for the two cases in which the cross term vanished:

(A) If O' is fixed:

$$T = \frac{1}{2}\vec{\omega}\cdot I_{O'}\vec{\omega}$$

(B) If O' is the COM:

$$T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\vec{\omega}\cdot I_{COM}\vec{\omega}$$

2.6.2 Continuous version of I

If we regard the rigid body as a continuum,

$$\boxed{I_{ab} = \sum_i m_i\left(\delta_{ab}\vec{r}_i^{\,2} - r_{ia}r_{ib}\right) \quad\rightarrow\quad \int \underbrace{d^3x\,\rho(\vec{x})}_{dm\text{:mass element}}\left(\delta_{ab}\vec{x}^2 - x_ax_b\right)}$$

Here $\rho(\vec{x})$ is the mass density function for the rigid body.

2.7 What frame do we use?

The vector ω and the tensor I are defined independently coordinate frames and we can compute their components in any frame. This is easier in certain frames than in others:

- in S_I the $\vec{r}_i$ are time-dependent
- in S_{II} the $\vec{r}_i$ are time-*independent*, so we will pick this one.

Henceforth, x, y, z will be taken to be in the body-fixed frame S_{II}. The inertia tensor becomes a purely geometric quantity inherent to the rigid body. It depends on the mass distribution of the rigid body.