SPA5304 Physical Dynamics Lecture 13

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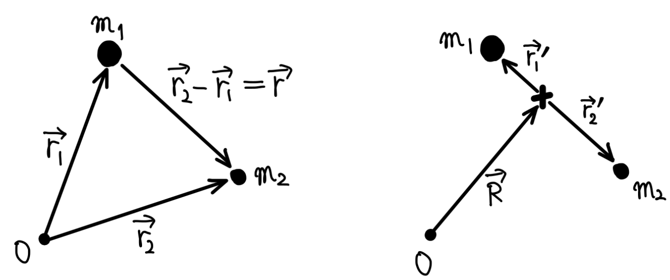
1 The Central Force Problem

As an important application of the Lagrangian formulation of mechanics, let us study the two-body problem with central forces.

1.1 Reduction to a one-body problem

Consider a system of two particles with masses m_1 and m_2.



- $\vec{r}_1$, $\vec{r}_2$ $\Rightarrow$ there are 6 degrees of freedom
- As generalized coordinates, take:
  - COM position $\vec{R}$
  - difference vector $\vec{r} \equiv \vec{r}_2 - \vec{r}_1$

What is the Lagrangian?

Recall the dcomposition of T in many-particle systems (see the end of Lecture 6):

$$T = \underbrace{\frac{1}{2}M\dot{\vec{R}}^2}_{\text{COM}} + \underbrace{\sum_{i=1}^{N}\frac{1}{2}m_i\dot{\vec{r}}_i{}'^2}_{\text{relative to COM}}$$

For two particles,

$$\vec{r}_1{}' = \vec{r}_1 - \vec{R} = \vec{r}_1 - \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} = -\frac{m_2(\vec{r}_2 - \vec{r}_1)}{m_1+m_2} = -\frac{m_2}{m_1+m_2}\vec{r}$$

$$\vec{r}_2{}' = \vec{r}_2 - \vec{R} = \vec{r}_2 - \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} = \frac{m_1(\vec{r}_2 - \vec{r}_1)}{m_1+m_2} = \frac{m_1}{m_1+m_2}\vec{r}$$

Thus,

$$T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}m_1\left(-\frac{m_2}{m_1+m_2}\dot{\vec{r}}\right)^2 + \frac{1}{2}m_2\left(\frac{m_1}{m_1+m_2}\dot{\vec{r}}\right)^2 = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\dot{\vec{r}}^2$$

and we get

$$\boxed{T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2}$$

where $\mu \equiv \frac{m_1 m_2}{m_1+m_2}$ is the **reduced mass**.

Assume that the potential depends only on the relative position: $V = V(\vec{r})$. Then,

$$\boxed{L_{\text{total}} = \underbrace{\frac{1}{2}M\dot{\vec{R}}^2}_{L_{\text{COM}}(\dot{\vec{R}})} + \underbrace{\frac{1}{2}\mu\dot{\vec{r}}^2 - V(\vec{r})}_{L_{\text{rel}}(\vec{r},\dot{\vec{r}})}}$$

Note that the COM motion and the relative motion *decouple* from each other.

Explicitly,

- E-L equation for $\vec{R}$:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\vec{R}}} = \frac{\partial L}{\partial\vec{R}} = 0 \qquad \Rightarrow \qquad M\ddot{\vec{R}} = 0 : \quad \text{trivial inertial motion}$$

- E-L equation for $\vec{r}$:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\vec{r}}} = \frac{\partial L}{\partial\vec{r}} = 0 \qquad \Rightarrow \qquad \mu\ddot{\vec{r}} = -\frac{\partial}{\partial\vec{r}}V(\vec{r}) : \quad \text{motion of a particle with Lagrangian } L_{\text{rel}}$$

The two-body problem thus reduces to a one-body problem. The #DoF has been reduced to 3.

1.2 Central force

Now consider just the relative motion described by $\vec{r}$.

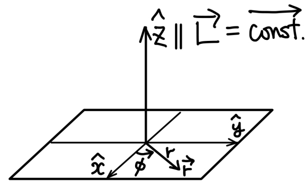
Assume $V = V(r)$ where $r \equiv |\vec{r}|$.

- The force is central: $\vec{F} || \vec{r}$

$$\vec{F} = -\frac{\partial V}{\partial\vec{r}} = -V'(r)\hat{r}$$

- Angular momentum $\vec{L}$ is conserved
- Motion is planar (since $\vec{r}$ is in a plane perpendicular to $\vec{L}$). Thus, the #DoF is reduced to 2.

Take polar coordinates (r, ϕ) on the plane of motion.



$$\boxed{L = T - V = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)}$$

The Euler-Lagrange equation for ϕ:

$$\frac{d}{dt}\underbrace{\frac{\partial L}{\partial \dot{\phi}}}_{p_\phi} = \frac{\partial L}{\partial \phi} = 0 \qquad \Rightarrow \qquad \dot{p}_\phi = 0$$

$$p_\phi = \mu r^2 \dot{\phi} = \text{const.} \equiv l \tag{1}$$

Claim: $p_\phi = L_z$.

Proof:

$$\vec{r} = (r\cos\phi,\, r\sin\phi,\, 0)$$
$$\dot{\vec{r}} = (\dot{r}\cos\phi - r\sin\phi\,\dot{\phi},\, \dot{r}\sin\phi + r\cos\phi\,\dot{\phi},\, 0)$$

Let's take the cross-product,

$$\vec{r} \times \dot{\vec{r}} = (0,\, 0,\, r^2\dot{\phi})$$
$$\vec{L} = \vec{r} \times \mu\dot{\vec{r}} = (0,\, 0,\, \mu r^2\dot{\phi})$$

so indeed $L_z = p_\phi$.

1.3 Reduction to one-dimensional problem

Energy is conserved,

$$E = T + V = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + V(r)$$

using $\dot{\phi} = \frac{l}{\mu r^2}$ from eqn. (1),

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\left(\frac{l}{\mu r^2}\right)^2 + V(r) = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + V(r)$$

or

$$\boxed{E = \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r) \qquad V_{\text{eff}}(r) = V(r) + \frac{l^2}{2\mu r^2}}$$

V_{eff} is the **effective potential**.

- This is the expression for the energy of a particle in one dimension with potential V_{eff}.
- The 2d problem has been reduced to a 1d problem (#DoF: $6 \to 3 \to 2 \to 1$)
- The extra "force" is

$$-\frac{d}{dr}\left(\frac{l^2}{2\mu r^2}\right) = \frac{l^2}{\mu r^3}$$

This is nothing but the centrifugal force

$$F_{\text{cf}} = \frac{\mu v^2}{r}, \quad v = r\dot{\phi} = \frac{l}{\mu r} \quad \Rightarrow \quad F_{\text{cf}} = \frac{l^2}{\mu r^3}$$

The Newton equation is

$$\boxed{\mu\ddot{r} = \frac{l^2}{\mu r^3} - V'(r)}$$

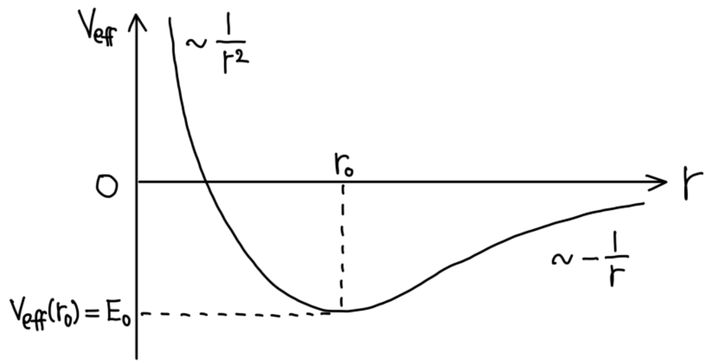
Note: It would have been incorrect to replace $\dot{\phi}$ in the Lagrangian by $\dot{\phi} = \frac{l}{\mu r^2}$ as we did in the formula for the energy. This would lead to a wrong effective potential $V_{\text{eff}}^{\text{WRONG}}(r) = V(r) - \frac{l^2}{2\mu r^2}$ which has the wrong sign for the second term. This is because the Lagrangian formulation assumes that the dynamical variables are independent (see p. 140 of Hand & Finch).

2 The Kepler Problem

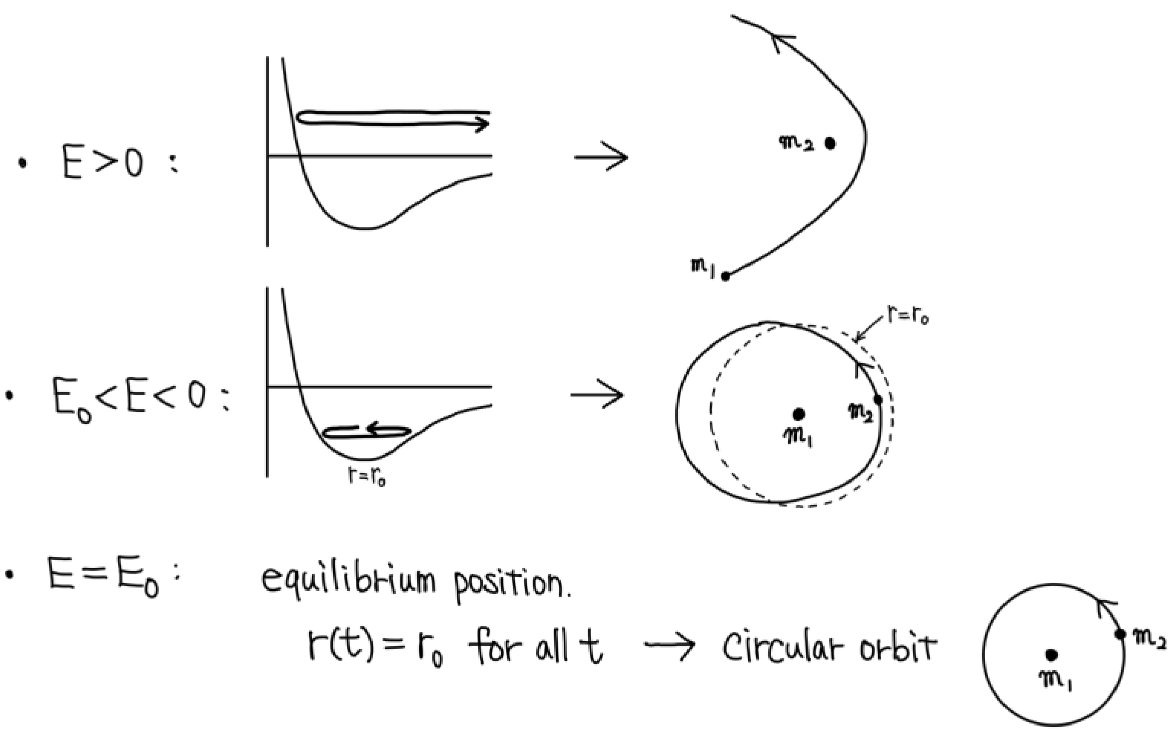
When the central force is the gravitational force, we have the Kepler potential:

$$V(r) = -\frac{Gm_1m_2}{r} \equiv -\frac{k}{r} \qquad\qquad k \equiv Gm_1m_2 > 0$$

$$\boxed{V_{\text{eff}} = \frac{l^2}{2\mu r^2} - \frac{k}{r}}$$



- A qualitative analysis of motion:



- What are r_0 and E_0? At the equilibrium position $r = r_0$, the total force is zero:

$$F = -V'_{\text{eff}}(r_0) = -\frac{l^2}{\mu r_0^3} + \frac{k}{r_0^2} = 0$$

which gives

$$\boxed{r_0 = \frac{l^2}{\mu k}}$$

Plugging this result back into the effective potential, we get

$$E_0 = \frac{l^2}{2\mu r_0^2} - \frac{k}{r_0} = -\frac{1}{2}\frac{\mu k^2}{l^2}$$

$$\boxed{E_0 = -\frac{\mu k^2}{2l^2} < 0}$$

2.1 The orbits

Let us now solve for the orbits. They will be a function $r = r(\phi)$ (there is no need for t). First of all, we have a conserved energy:

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{l}{2\mu r^2} - \frac{k}{r}$$

From this we can express $\dot{r}$:

$$\dot{r} = \sqrt{\frac{2}{\mu}\left(E - \frac{l^2}{2\mu r^2} + \frac{k}{r}\right)}$$

We also have

$$\dot{\phi} = \frac{l}{\mu r^2}$$

We can eliminate time if we take the ratio:

$$\frac{d\phi}{dr} = \frac{\left(\frac{d\phi}{dt}\right)}{\left(\frac{dr}{dt}\right)} = \frac{\dot{\phi}}{\dot{r}} = \frac{l}{\mu r^2}\frac{1}{\sqrt{\frac{2}{\mu}\left(E - \frac{l^2}{2\mu r^2} + \frac{k}{r}\right)}}$$

Integrating this gives

$$\phi(r_2) - \phi(r_1) = \int_{r_1}^{r_2} \frac{l}{\mu r^2}\frac{1}{\sqrt{\frac{2}{\mu}\left(E - \frac{l^2}{2\mu r^2} + \frac{k}{r}\right)}}$$

We have to do this integral. Complete the square for $1/r$ and use the expressions for r_0 and E_0 to get

$$\phi(r_2) - \phi(r_1) = \int_{r_1}^{r_2} \frac{dr}{r^2}\frac{1}{\sqrt{\frac{2\mu}{l^2}\left(E - E_0\right) - \left(\frac{1}{r} - \frac{1}{r_0}\right)^2}}$$

Denote $x \equiv \frac{1}{r} - \frac{1}{r_0}$. Then $dx = -\frac{dr}{r^2}$ and we have

$$\phi(r_2) - \phi(r_1) = -\int_{x_1}^{x_2} \frac{dx}{\sqrt{\frac{2\mu}{l^2}\left(E - E_0\right) - x^2}} = \arccos\left(\frac{x}{\sqrt{\frac{2\mu}{l^2}\left(E - E_0\right)}}\right)\Bigg|_{x_1}^{x_2} = \arccos\left(\frac{\frac{1}{r} - \frac{1}{r_0}}{\sqrt{\frac{2\mu}{l^2}\left(E - E_0\right)}}\right)\Bigg|_{r_1}^{r_2}$$

Let us now take the zero point of ϕ such that $\phi(r_1) = 0$. Also, write r instead of r_2. We get the equation

$$\sqrt{\frac{2\mu}{l^2}\left(E - E_0\right)} \cos\phi = \frac{1}{r} - \frac{1}{r_0} \tag{2}$$

Here

$$\frac{2\mu}{l^2} E_0 = \frac{2\mu}{l^2}\left(-\frac{\mu k^2}{2l^2}\right) = -\frac{1}{r_0^2}$$

So

$$\frac{2\mu}{l^2} = -\frac{1}{r_0^2 E_0}$$

Using this, the square root can be re-written as

$$\sqrt{\frac{2\mu}{l^2}\left(E - E_0\right)} = \sqrt{-\frac{1}{r_0^2 E_0}\left(E - E_0\right)} = \frac{1}{r_0}\sqrt{1 - \frac{E}{E_0}}$$

and equation (2) becomes

$$r = \frac{r_0}{1 + \varepsilon \cos\phi}$$

where the **eccentricity** ε is defined as

$$\varepsilon \equiv \sqrt{1 - \frac{E}{E_0}}$$

Note that E_0 is negative.