SPA5304 Physical Dynamics Lecture 11-12

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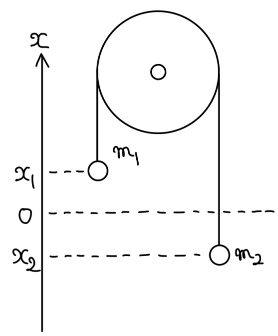
1 Lagrangian mechanics

1.1 Example: Atwood's machine



Invented by George Atwood in 1784 to verify the mechanical law of motion with constant acceleration.

- Constraint: $x_1 + x_2 = 0$
- 1 degree of freedom
- generalized coordinate: $x \equiv x_1 = -x_2$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 = \frac{1}{2}(m_1 + m_2)\dot{x}^2$$

$$V = m_1 g x_1 + m_2 g x_2 = (m_1 - m_2) g x$$

$$L = T - V = \frac{1}{2}(m_1 + m_2)\dot{x}^2 - (m_1 - m_2) g x$$

The Euler-Lagrange equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = (m_1 + m_2)\ddot{x} + (m_1 - m_2) g = 0$$

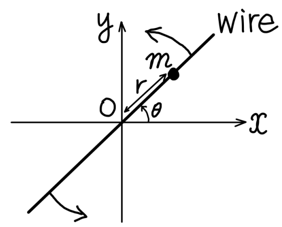
and thus

$$\ddot{x} = -\frac{m_1 - m_2}{m_1 + m_2} g$$

No constraint forces (tension) appear.

1.2 Example: A bead sliding on a uniformly rotating wire

An example of time-dependent constraints.



- Constraint: $\theta = \omega t$ (this is time-dependent!)
- 1 degree of freedom
- generalised coordinate: r

$$\begin{cases} x = r\cos\omega t \\ y = r\sin\omega t \end{cases}$$

Using the expression for T in polar coordinates,

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\omega^2)$$

Note that this is not quadratic in $\dot{r}$.

If there are no other forces, $L = T$. The E-L equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = m\ddot{r} - mr\omega^2 = 0$$

$$\ddot{r} = r\omega^2$$

$$r = r_0 e^{\omega t}$$

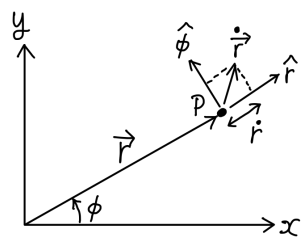
This means that the bead moves exponentially outward.

- Again no constraint force appears in the equation.
- The constraint force is *not* perpendicular to the motion, and thus it does work on the bead (energy is not conserved).

2 Curvilinear coordinates

Let us recall a few examples for curvilinear coordinate systems. These will be useful as generalised coordinates.

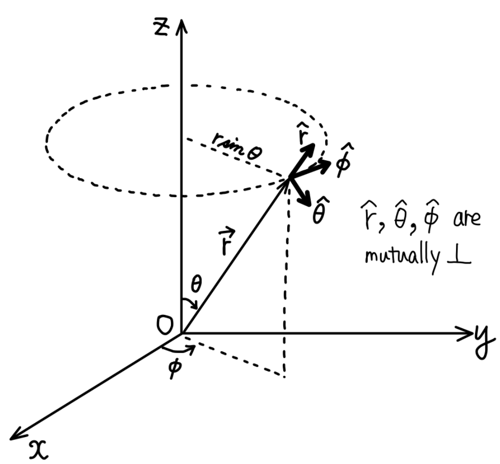
2.1 Polar coordinates



We have already seen this example in Lecture 9.

$$T = \frac{1}{2} m \dot{\vec{r}}^2 = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2)$$

2.2 Spherical coordinates



$$\vec{r} = (x,\, y,\, z) = r(\sin\theta\cos\phi,\, \sin\theta\sin\phi,\, \cos\theta)$$

$$\begin{aligned} \theta \in [0, \pi] \quad &: \quad \text{polar angle (colatitude)} \\ \phi \in [0, 2\pi] \quad &: \quad \text{azimuthal angle} \end{aligned}$$

- Components of the hatted vectors are

$$\hat{r} = (\sin\theta\cos\phi,\, \sin\theta\sin\phi,\, \cos\theta)$$

Take the θ derivative of this to get

$$\hat{\theta} = (\cos\theta\cos\phi,\, \cos\theta\sin\phi,\, -\sin\theta)$$

Finally,

$$\hat{\phi} = (-\sin\phi,\, \cos\phi,\, 0)$$

One can check that the vectors are orthonormal:

$$\hat{r}^2 = \hat{\theta}^2 = \hat{\phi}^2 = 1$$

$$\hat{r}\cdot\hat{\theta} = \hat{r}\cdot\hat{\phi} = \hat{\theta}\cdot\hat{\phi} = 0$$

and that $\hat{r}\cdot(\hat{\theta}\times\hat{\phi}) = +1$, i.e. they form a right-handed frame.

- In the kinetic term $\dot{\vec{r}}$ shows up which we need to compute

$$\dot{\vec{r}} = \frac{\partial\vec{r}}{\partial r}\dot{r} + \frac{\partial\vec{r}}{\partial\theta}\dot{\theta} + \frac{\partial\vec{r}}{\partial\phi}\dot{\phi}$$

We can find expressions for the partial derivatives using the hatted vectors

$$\frac{\partial\vec{r}}{\partial r} = \hat{r}, \qquad \frac{\partial\vec{r}}{\partial\theta} = r\hat{\theta}, \qquad \frac{\partial\vec{r}}{\partial\phi} = r\sin\theta\,\hat{\phi}$$

Plugging these back in gives

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\,\dot{\phi}\hat{\phi}$$

and

$$\boxed{T = \frac{m}{2}\dot{\vec{r}}^2 = \frac{m}{2}\left(\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)\right)}$$

- The conjugate momenta are

$$p_r = \frac{\partial L}{\partial\dot{r}} = \frac{\partial T}{\partial\dot{r}} = m\dot{r}$$

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = \frac{\partial T}{\partial\dot{\theta}} = mr^2\dot{\theta}$$

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = \frac{\partial T}{\partial\dot{\phi}} = mr^2\sin^2\theta\,\dot{\phi}$$

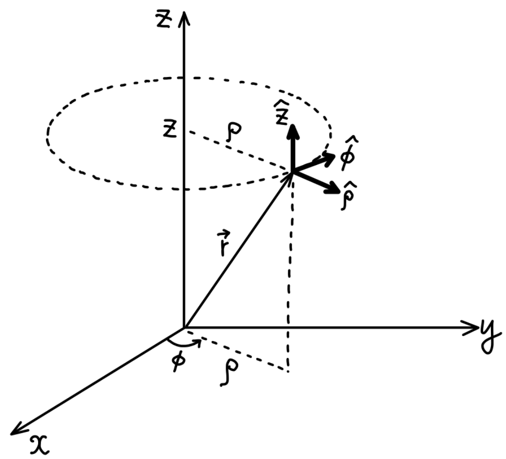
- Angular momentum (about origin)

$$\vec{L} = m\vec{r}\times\dot{\vec{r}} = mr\hat{r}\times(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\,\dot{\phi}\hat{\phi}) = mr^2(\dot{\theta}\underbrace{\hat{r}\times\hat{\theta}}_{\hat{\phi}} + \sin\theta\,\dot{\phi}\underbrace{\hat{r}\times\hat{\phi}}_{-\hat{\theta}}) = mr^2\dot{\theta}\hat{\phi} - mr^2\sin\theta\,\dot{\phi}\hat{\theta}$$

$$L_z = \vec{L}\cdot\hat{z} = mr^2\dot{\theta}\underbrace{\hat{\phi}\cdot\hat{z}}_{0} - mr^2\sin\theta\,\dot{\phi}\underbrace{\hat{\theta}\cdot\hat{z}}_{-\sin\theta} = mr^2\sin^2\theta\,\dot{\phi}$$

So $L_z = p_\phi$ and it is conserved if ϕ is cyclic.

2.3 Cylindrical coordinates



$$\vec{r} = (\rho\cos\phi,\ \rho\sin\phi,\ z)$$

$$\dot{\vec{r}} = \frac{\partial\vec{r}}{\partial\rho}\dot{\rho} + \frac{\partial\vec{r}}{\partial\phi}\dot{\phi} + \frac{\partial\vec{r}}{\partial z}\dot{z} = \hat{\rho}\dot{\rho} + \rho\hat{\phi}\dot{\phi} + \hat{z}\dot{z}$$

where

$$\begin{aligned} \hat{r} = &\quad \tfrac{\partial\vec{r}}{\partial\rho} = &\quad (\cos\phi,\ \sin\phi,\ 0) \\ \hat{\phi} = &\quad \tfrac{1}{\rho}\tfrac{\partial\vec{r}}{\partial\phi} = &\quad (-\sin\phi,\ \cos\phi,\ 0) \\ \hat{z} = &\quad \tfrac{\partial\vec{r}}{\partial z} = &\quad (0,\ 0,\ 1) \end{aligned}$$

These three vectors form a right-handed orthonormal basis.

$$\boxed{T = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)}$$

The conjugate momenta are

$$p_\rho = \frac{\partial L}{\partial\dot{\rho}} = m\dot{\rho}$$

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = m\rho^2\dot{\phi}$$

$$p_z = \frac{\partial L}{\partial\dot{z}} = m\dot{z}$$

$$\vec{L} = m\vec{r}\times\dot{\vec{r}} = m(\rho\hat{\rho} + z\hat{z})\times(\hat{\rho}\dot{\rho} + \rho\hat{\phi}\dot{\phi} + \hat{z}\dot{z}) = m(\rho^2\dot{\phi}\underbrace{\hat{\rho}\times\hat{\phi}}_{\hat{z}} + \rho\dot{z}\underbrace{\hat{\rho}\times\hat{z}}_{-\hat{\phi}} + z\dot{\rho}\underbrace{\hat{z}\times\hat{\rho}}_{\hat{\phi}} + z\rho\dot{\phi}\underbrace{\hat{z}\times\hat{\phi}}_{-\hat{\rho}})$$

$$\vec{L} = m\rho^2\dot{\phi}\hat{z} + m(z\dot{\rho} - \rho\dot{z})\hat{\phi} - m\rho z\dot{\phi}\hat{\rho}$$

$$L_z = \vec{L}\cdot\hat{z} = m\rho^2\dot{\phi} = p_\phi$$

L_z is conserved if ϕ is cyclic.

Let us look at some examples of Lagrangian mechanics using these coordinate systems.

2.4 Example: A free particle in spherical coordinates

$$L = T = \frac{m}{2}(\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2))$$

The Euler-Lagrange equations

- for r:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r}$$

$$\frac{d}{dt}(m\dot{r}) = m\ddot{r} = mr(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)$$

- for θ:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$$

$$\frac{d}{dt}(mr^2\dot{\theta}) = mr^2\sin\theta\cos\theta\,\dot{\phi}^2$$

- for ϕ:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} = 0 \qquad \text{cyclic coordinate}$$

$$\frac{d}{dt}(mr^2\sin^2\theta\,\dot{\phi}) = 0 \quad \Rightarrow \quad p_\phi = mr^2\sin^2\theta\,\dot{\phi} = \text{const.}$$

2.5 Example: A free particle in cylindrical coordinates

$$L = T = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)$$

The Euler-Lagrange equations

- for z:

$$\frac{\partial L}{\partial z} = 0 \qquad \text{cyclic coordinate}$$

thus the linear momentum along z is constant:

$$p_z = m\dot{z} = \text{const.}$$

- for ρ:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\rho}} = \frac{\partial L}{\partial \rho}$$

$$m\ddot{\rho} = \underbrace{m\rho\dot{\phi}^2}_{\text{centrifugal force}}$$

- for ϕ:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} = 0 \qquad \text{cyclic coordinate}$$

$$p_\phi = m\rho^2\dot{\phi} = \text{const.} \equiv l$$

Plugging this into the E-L equation for ρ,

$$m\ddot{\rho} = m\rho\left(\frac{l}{m\rho^2}\right)^2 = \frac{l^2}{m\rho^3}$$

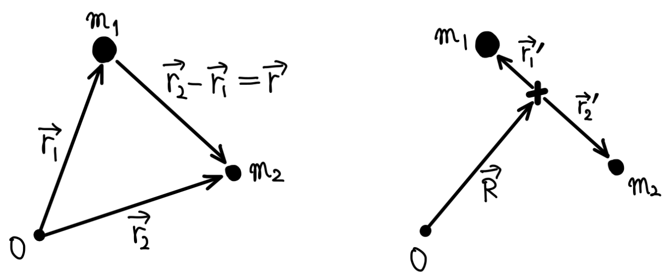
The RHS is the centrifugal force.

3 The Central Force Problem

As an important application of the Lagrangian formulation of mechanics, let us study the two-body problem with central forces.

3.1 Reduction to a one-body problem

Consider a system of two particles with masses m_1 and m_2.



- $\vec{r}_1$, $\vec{r}_2$ $\Rightarrow$ there are 6 degrees of freedom
- As generalized coordinates, take:
  - COM position $\vec{R}$
  - difference vector $\vec{r} \equiv \vec{r}_2 - \vec{r}_1$

  What is the Lagrangian?

  Recall the dcomposition of T in many-particle systems (see the end of Lecture 6):

$$T = \underbrace{\frac{1}{2}M\dot{\vec{R}}^2}_{\text{COM}} + \underbrace{\sum_{i=1}^{N}\frac{1}{2}m_i\dot{\vec{r}}_i'^2}_{\text{relative to COM}}$$

For two particles,

$$\vec{r}_1{}' = \vec{r}_1 - \vec{R} = \vec{r}_1 - \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = -\frac{m_2(\vec{r}_2 - \vec{r}_1)}{m_1 + m_2} = -\frac{m_2}{m_1 + m_2}\vec{r}$$

$$\vec{r}_2{}' = \vec{r}_2 - \vec{R} = \vec{r}_2 - \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = \frac{m_1(\vec{r}_2 - \vec{r}_1)}{m_1 + m_2} = \frac{m_1}{m_1 + m_2}\vec{r}$$

Thus,

$$T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}m_1\left(-\frac{m_2}{m_1+m_2}\dot{\vec{r}}\right)^2 + \frac{1}{2}m_2\left(\frac{m_1}{m_1+m_2}\dot{\vec{r}}\right)^2 = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\dot{\vec{r}}^2$$

and we get

$$\boxed{T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2}$$

where $\mu \equiv \frac{m_1 m_2}{m_1+m_2}$ is the **reduced mass**.

Assume that the potential depends only on the relative position: $V = V(\vec{r})$. Then,

$$\boxed{L_{\text{total}} = \underbrace{\frac{1}{2}M\dot{\vec{R}}^2}_{L_{\text{COM}}(\dot{\vec{R}})} + \underbrace{\frac{1}{2}\mu\dot{\vec{r}}^2 - V(\vec{r})}_{L_{\text{rel}}(\vec{r},\dot{\vec{r}})}}$$

Note that the COM motion and the relative motion *decouple* from each other.

Explicitly,

- E-L equation for $\vec{R}$:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\vec{R}}} = \frac{\partial L}{\partial\vec{R}} = 0 \qquad \Rightarrow \qquad M\ddot{\vec{R}} = 0 : \quad \text{trivial inertial motion}$$

- E-L equation for $\vec{r}$:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\vec{r}}} = \frac{\partial L}{\partial\vec{r}} = 0 \qquad \Rightarrow \qquad \mu\ddot{\vec{r}} = -\frac{\partial}{\partial\vec{r}}V(\vec{r}) : \quad \text{motion of a particle with Lagrangian } L_{\text{rel}}$$

The two-body problem thus reduces to a one-body problem. The #DoF has been reduced to 3.

3.2 Central force

Now consider just the relative motion described by $\vec{r}$.

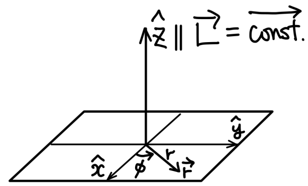
Assume $V = V(r)$ where $r \equiv |\vec{r}|$.

- The force is central: $\vec{F}||\vec{r}$

$$\vec{F} = -\frac{\partial V}{\partial\vec{r}} = -V'(r)\hat{r}$$

- Angular momentum $\vec{L}$ is conserved
- Motion is planar (since $\vec{r}$ is in a plane perpendicular to $\vec{L}$). Thus, the #DoF is reduced to 2.

Take polar coordinates (r, ϕ) on the plane of motion.



$$L = T - V = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)$$

The Euler-Lagrange equation for ϕ:

$$\frac{d}{dt}\underbrace{\frac{\partial L}{\partial \dot{\phi}}}_{p_\phi} = \frac{\partial L}{\partial \phi} = 0 \qquad \Rightarrow \qquad \dot{p}_\phi = 0$$

$$p_\phi = \mu r^2 \dot{\phi} = \text{const.} \equiv l \tag{1}$$

Claim: $p_\phi = L_z$.

Proof:

$$\vec{r} = (r\cos\phi,\, r\sin\phi,\, 0)$$

$$\dot{\vec{r}} = (\dot{r}\cos\phi - r\sin\phi\,\dot{\phi},\, \dot{r}\sin\phi + r\cos\phi\,\dot{\phi},\, 0)$$

Let's take the cross-product,

$$\vec{r} \times \dot{\vec{r}} = (0,\, 0,\, r^2\dot{\phi})$$

$$\vec{L} = \vec{r} \times \mu\dot{\vec{r}} = (0,\, 0,\, \mu r^2\dot{\phi})$$

so indeed $L_z = p_\phi$.

3.3 Reduction to one-dimensional problem

Energy is conserved,

$$E = T + V = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + V(r)$$

using $\dot{\phi} = \frac{l}{\mu r^2}$ from eqn. (1),

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\left(\frac{l}{\mu r^2}\right)^2 + V(r) = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + V(r)$$

or

$$\boxed{E = \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r) \qquad V_{\text{eff}}(r) = V(r) + \frac{l^2}{2\mu r^2}}$$

V_{eff} is the **effective potential**.

- This is the expression for the energy of a particle in one dimension with potential V_{eff}.
- The 2d problem has been reduced to a 1d problem (#DoF: $6 \to 3 \to 2 \to 1$)
- The extra "force" is

$$-\frac{d}{dr}\left(\frac{l^2}{2\mu r^2}\right) = \frac{l^2}{\mu r^3}$$

This is nothing but the centrifugal force

$$F_{\text{cf}} = \frac{\mu v^2}{r}, \quad v = r\dot{\phi} = \frac{l}{\mu r} \quad \Rightarrow \quad F_{\text{cf}} = \frac{l^2}{\mu r^3}$$

The Newton equation is

$$\boxed{\mu\ddot{r} = \frac{l^2}{\mu r^3} - V'(r)}$$

Note: It would have been incorrect to replace $\dot{\phi}$ in the Lagrangian by $\dot{\phi} = \frac{l}{\mu r^2}$ as we did in the formula for the energy. This would lead to a wrong effective potential $V_{\text{eff}}^{\text{WRONG}}(r) = V(r) - \frac{l^2}{2\mu r^2}$ which has the wrong sign for the second term. This is because the Lagrangian formulation assumes that the dynamical variables are independent (see p. 140 of Hand & Finch).