SPA5304 Physical Dynamics Lecture 10

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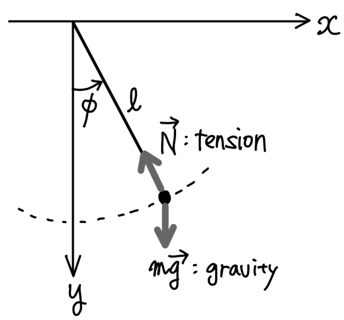
1 Lagrangian mechanics

1.1 Example: The planar pendulum



The purpose of this section is to:

(i) see the power of the Lagrangian approach

(ii) demonstrate that the variational principle works for constrained systems

1.1.1 Newtonian approach

$$\dot{\vec{p}} = m\vec{g} + \vec{N}$$

Components:

$$\begin{cases} m\ddot{x} = -N\sin\phi \\ m\ddot{y} = mg - N\cos\phi \end{cases}$$

where $N \equiv |\vec{N}|$.

$\vec{N}$ is a **constraint force**. It is perpendicular to the motion $\Rightarrow$ does no work $\Rightarrow$ energy is conserved.

Relation between the Cartesian coordinates (x, y) and ϕ:

$$\begin{cases} x = l\sin\phi \\ y = l\cos\phi \end{cases} \Rightarrow \begin{cases} \dot{x} = l(\cos\phi)\dot{\phi} \\ \dot{y} = -l(\sin\phi)\dot{\phi} \end{cases} \Rightarrow \begin{cases} \ddot{x} = l(+\cos\phi\,\ddot{\phi} - \sin\phi\,\dot{\phi}^2) \\ \ddot{y} = l(-\sin\phi\,\ddot{\phi} - \cos\phi\,\dot{\phi}^2) \end{cases}$$

Plug these into the equation of motion:

$$\begin{cases} ml(\cos\phi\,\ddot{\phi} - \sin\phi\,\dot{\phi}^2) = -N\sin\phi \\ ml(-\sin\phi\,\ddot{\phi} - \cos\phi\,\dot{\phi}^2) = mg - N\cos\phi \end{cases}$$

Multiply the first equation by $\cos\phi$ and the second equation by $\sin\phi$, then subtract the two equations from each other to eliminate N. We get

$$\ddot{\phi} = -\frac{g}{l}\sin\phi$$

N can be expressed

$$N = \frac{ml(\cos\phi\,\ddot{\phi} - \sin\phi\,\dot{\phi}^2)}{-\sin\phi} = \underbrace{mg\cos\phi}_{\text{counter gravity}} + \underbrace{ml\dot{\phi}^2}_{\text{centripetal force}}$$

We had to introduce a constraint force $\vec{N}$. We also needed the EOMs for x and y to derive an EOM for ϕ.

1.1.2 Lagrangian approach

There is one (holonomic) constraint: $r = l$. Thus, there is 1 degree of freedom. We can use ϕ as the generalised coordinate.

$$\left\{\begin{array}{l} x = l\sin\phi \\ y = l\cos\phi \end{array}\right. \quad \Rightarrow \left\{\begin{array}{l} \dot{x} = l(\cos\phi)\dot{\phi} \\ \dot{y} = -l(\sin\phi)\dot{\phi} \end{array}\right.$$

- **Kinetic energy**

  Plug the above expressions for $\dot{x}$ and $\dot{y}$ in the Cartesian formula:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}ml^2\dot{\phi}^2$$

Alternatively, we can start with the kinetic energy expressed in polar coordinates, and then set $r = l$, $\dot{r} = 0$:

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) = \frac{1}{2}ml^2\dot{\phi}^2$$

- **Potential energy**

$$V = -mgy = -mgl\cos\phi \qquad \text{(the } y \text{ axis is pointing downwards!)}$$

- **Lagrangian**

$$\boxed{L = T - V = \frac{1}{2}ml^2\dot{\phi}^2 + mgl\cos\phi}$$

The momentum conjugate to ϕ:

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ml^2\dot{\phi}$$

The Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi}$$

gives

$$ml^2\ddot{\phi} = -mgl\sin\phi$$

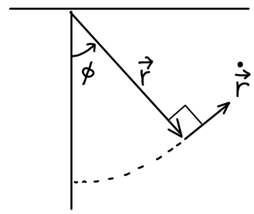
$$\boxed{\ddot{\phi} = -\frac{g}{l}\sin\phi}$$

- This derivation is much simpler and more straightforward than the old Newtonian method:

(i) $\vec{N}$ never appeared.

(ii) Directly got the equation of motion for ϕ.

- The energy $E = T + V$ is conserved. (Check it)



- p_ϕ is angular momentum

$$\vec{L} = \vec{r} \times m\dot{\vec{r}}$$
$$|\vec{L}| = l \cdot m\sqrt{\dot{x}^2 + \dot{y}^2} = l \cdot ml|\dot{\phi}| = |p_\phi|$$

Therefore

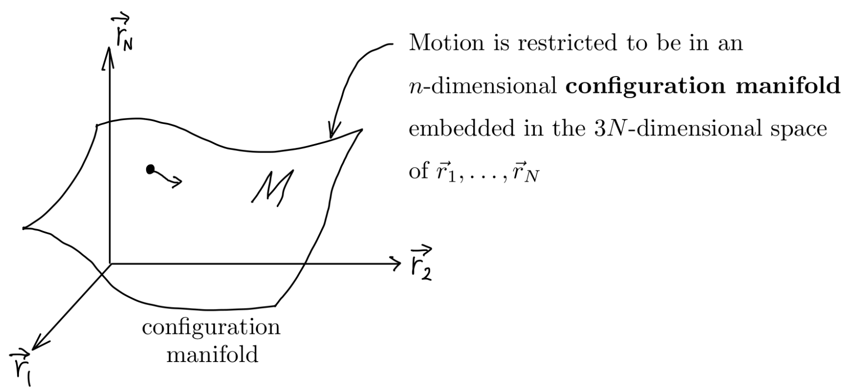
$$|p_\phi| = |\vec{L}|$$

2 Constraints

- **Holonomic constraints**

  Constraints that can be written in the form of

$$\left.\begin{array}{l} f_1(\vec{r}_1,\ldots,\vec{r}_N,t)=0 \\ f_2(\vec{r}_1,\ldots,\vec{r}_N,t)=0 \\ \vdots \\ f_h(\vec{r}_1,\ldots,\vec{r}_N,t)=0 \end{array}\right\} h \text{ equations} \quad \to \quad \text{reduce the \#DoF from } 3N \text{ to } 3N-h\equiv n$$



  The constraint equations can be solved in therms of n independent variables

$$\vec{q}=(q_1,\ldots,q_n)$$

These are the generalised coordinates.

$$\left\{\begin{array}{l} \vec{r}_1=\vec{r}_1(q_1,\ldots,q_n) \\ \vdots \\ \vec{r}_N=\vec{r}_N(q_1,\ldots,q_n) \end{array}\right.$$

- **Non-holonomic constraints**

  These are all other types of constraints which are not holonomic.

2.1 Examples for holonomic constraints

2.1.1 The planar pendulum

The constraint equation is

$$f = x^2 + y^2 - l^2 = 0$$

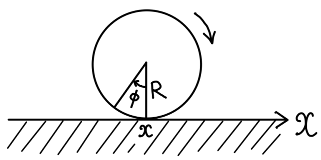
This can be solved using the generalised coordinate ϕ as

$$x = l \sin\phi \qquad y = l\cos\phi$$

2.1.2 Rigid bodies

$$|\vec{r}_i - \vec{r}_j| - c_{ij} = 0$$

2.1.3 A disk rolling without slipping in one dimension



No slipping means $dx = Rd\phi$, or if we divide both sides by dt,

$$\dot{x} = R\dot{\phi} \tag{1}$$

This can be integrated to give

$$x = R\phi + \text{const.}$$

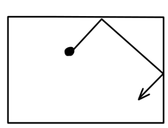
which is holonomic. We can use either x or ϕ as the generalised coordinate.

2.2 Examples for non-holonomic constraints

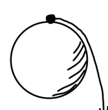
2.2.1 A disk rolling without slipping in higher dimensions

In higher dimensions, the analog of (1) cannot be integrated and the constraint is therefore non-holonomic (see Goldstein et al. p15-16).

2.2.2 Other examples



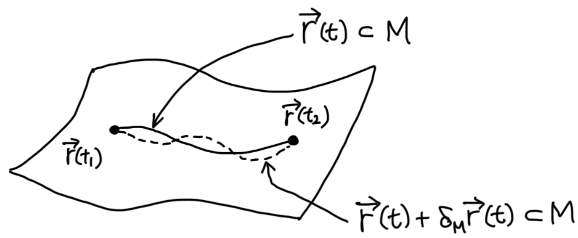
A particle in a box



A particle on a sphere

$r \geq R$

2.3 Variational calculus with constraints

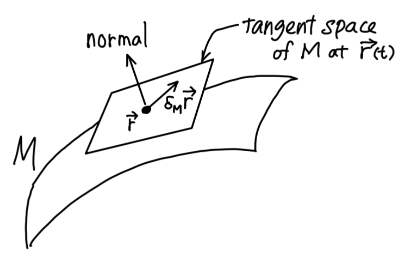


$$S = \int_{t_1}^{t_2} dt\, L(\vec{r}, \dot{\vec{r}})$$

Find the extremum as before. Now the variation is restricted to be in M:

$$0 = \delta_M S = \delta_M \int_{t_1}^{t_2} dt\, L(\vec{r}, \dot{\vec{r}}) = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial \vec{r}} \delta_M \vec{r} + \frac{\partial L}{\partial \dot{\vec{r}}} \delta_M \dot{\vec{r}} \right) = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial \vec{r}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{r}}} \right) \delta_M \vec{r}$$

This is the same result as in the unconstrained case, except we have $\delta_M \vec{r}$ instead of $\delta \vec{r}$. From this result we cannot conclude that the Euler-Lagrange equation inside the parentheses vanishes, because $\delta_M \vec{r}$ is not completely arbitrary: it is constrained to lie in the tangent space of M.



So the above result does not put any constraint on the normal component and we get the equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{r}}} - \frac{\partial L}{\partial \vec{r}} = \vec{N}$$

where $\vec{N}$ is a constraint force perependicular to M.

2.4 Procedure for solving problems with holonomic constraints

(i) Determine the configuration manifold (e.g. $x^2 + y^2 = l^2$) and introduce generalised coordinates on it:

$$\{q_i\}, \quad i = 1, 2, \ldots, n \quad \text{where } n \text{ is the \#DoF}$$

(ii) Re-express the kinetic energy T in terms of $\vec{q}$ and $\dot{\vec{q}}$.

If the constraints are time-independent and $\vec{r}_i = \vec{r}_i(\vec{q})$, then T is quadratic in $\dot{\vec{q}}$

$$T = \sum_{i,j} \frac{1}{2} a_{ij}(\vec{q}) \dot{q}_i \dot{q}_j$$

E.g. planar pendulum: $q = \phi$ and $T = \frac{1}{2} \underbrace{ml^2}_{a(\phi)} \dot{\phi}^2$

(iii) Construct the Lagrangian

$$L = T - V(\vec{q})$$

and solve the Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}, \qquad i = 1, 2, \ldots, n$$