SPA5304 Physical Dynamics Lecture 8-9

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23 January 2019

1 Lagrangian mechanics

1.1 Calculus of variations

In general, we will consider functionals which depend on y, $\dot{y}$, and x. They take the form

$$\boxed{F[y] = \int_{x_1}^{x_2} dx\, L\,(y(x),\, \dot{y}(x),\, x)}$$

Our previous example $l[y]$ had this form (L did not explicitly depend on $y(x)$).

Write $\delta y \equiv \eta$ to simplify notation. For an extremum we have

$$0 = \delta F \equiv F[y+\eta] - F[y] = \int_{x_1}^{x_2} dx\, (L(y+\eta, \dot{y}+\dot{\eta}, x) - L(y, \dot{y}, x)) \tag{1}$$

Taylor expansion of the first term: $L(y+\eta, \dot{y}+\dot{\eta}, x) = L + \frac{\partial L}{\partial y}\eta + \frac{\partial L}{\partial \dot{y}}\dot{\eta} + \mathcal{O}(\eta^2)$.

Note: here we are treating $y(x)$ and $\dot{y}(x)$ as independent symbols when we compute $\frac{\partial L}{\partial y}$ and $\frac{\partial L}{\partial \dot{y}}$.

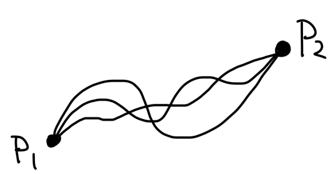
By plugging this into eqn. (1) we get

$$0 = \int_{x_1}^{x_2} dx \left(\frac{\partial L}{\partial y}\eta + \frac{\partial L}{\partial \dot{y}}\dot{\eta} \right)$$

The second term in the integrand can be written as: $\frac{\partial L}{\partial \dot{y}}\dot{\eta} = \frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}}\eta\right) - \left(\frac{d}{dx}\frac{\partial L}{\partial \dot{y}}\right)\eta$. Plugging this in gives

$$0 = \int_{x_1}^{x_2} dx \left(\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial \dot{y}} \right)\eta + \left[\frac{\partial L}{\partial \dot{y}}\eta \right]_{x_1}^{x_2}$$

When we vary the function, we keep the endpoints of the curve fixed:



The boundary conditions are:

$$\delta y(x_1) = \delta y(x_2) = 0$$

and thus the second term will vanish.

Apart from the endpoints, $\eta(x)$ is arbitrary. In order for the integral to vanish ($\delta F = 0$), the integrand must vanish for every x. This gives the Euler-Lagrange equation:

$$\boxed{\frac{d}{dx}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0}$$

This equation has to be satisfied if $F[y]$ is to be extremized.

1.2 Example: Length of a curve

Let's go back to our previous example. The length functional was

$$l[y] = \int_{x_1}^{x_2} dx\, L(y, \dot{y}, x)$$

where $L = \sqrt{1 + \dot{y}^2}$.

Let us now write down the Euler-Lagrange equation. We have

$$\frac{\partial L}{\partial y} = 0 \qquad \text{and} \qquad \frac{\partial L}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{1 + \dot{y}^2}}$$

Thus the Euler-Lagrange equation is

$$\frac{d}{dx}\frac{\dot{y}}{\sqrt{1 + \dot{y}^2}} = 0$$

Integrating this gives

$$\frac{\dot{y}}{\sqrt{1 + \dot{y}^2}} = \text{const}$$

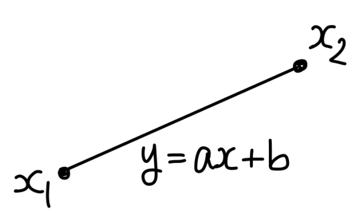
which is equivalent to

$$\dot{y} = \text{const} \equiv a$$

or

$$y = ax + b$$

which is a straight line! (We expected this of course.)



1.3 Many degrees of freedom

"Multi-variable" functionals:

$$F[y_1(x), y_2(x), \ldots, y_N(x)] \equiv F[\vec{y}] = \int_{x_1}^{x_2} dx\, L(\vec{y}(x),\, \dot{\vec{y}}(x),\, x)$$

The derivation we just saw applies again, as long as $y_i(x)$ are independent of each other:

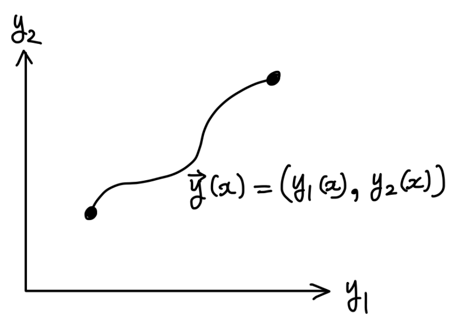
$$\boxed{\frac{d}{dx}\frac{\partial L}{\partial \dot{y}_i} - \frac{\partial L}{\partial y_i} = 0}$$

- Note that we have one Euler-Lagrange equation for each y_i $(i = 1, \ldots, N)$.
- Note that the endpoints have been fixed: $\eta_i(x_1) = \eta_i(x_2) = 0$ for all i.

1.4 Coordinate independence

We can think of $\vec{y}(x) = (y_1(x), y_2(x), \ldots, y_n(x))$ as describing a path parametrized by x in an n-dimensional space with coordinates $\vec{y}$.

The path that extremizes $F[\vec{y}]$ satisfies the E-L eqns.



The path that extremizes $F[\vec{y}]$ satisfies the E-L eqns.

- What if we change the coordinates $\vec{y} \to \vec{z} = \vec{z}(\vec{y})$?

This means that we change the functional $F[\vec{y}] \to F[\vec{z}]$ as well.

From the geometric picture it is clear that the path itself that extremizes F is independent of the coordinate system that we use to describe it. Thus, the Euler-Lagrange equation in any coordinate system should give the same path. This will be relvant in applications to mechanics: physics should be independent of particular choices of generalized coordinates.

Proof:

The Euler-Lagrange equation:

$$\boxed{\frac{d}{dx}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0}$$

Let's change coordinates: $y \to z$ so that $y = y(z,t)$. Note that y does not depend on $\dot{z}$.

Then the Lagrangian is

$$L = L(y(z,x),\, \dot{y}(z,\dot{z},x),\, x) \equiv \tilde{L}(z(x),\, \dot{z}(x),\, x)$$

We have

$$\dot{y} = \frac{\partial y}{\partial z}\dot{z} + \frac{\partial y}{\partial x}$$

This implies

$$\frac{\partial \dot{y}}{\partial \dot{z}} = \frac{\partial y}{\partial z} \tag{2}$$

We further have

$$\frac{\partial \tilde{L}}{\partial z} = \frac{\partial L}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial L}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial z} = \frac{\partial L}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial L}{\partial \dot{y}}\frac{d}{dx}\frac{\partial y}{\partial z}$$

and

$$\frac{\partial \tilde{L}}{\partial \dot{z}} = \frac{\partial L}{\partial y}\underbrace{\frac{\partial y}{\partial \dot{z}}}_{=0} + \frac{\partial L}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \dot{z}}$$

Using (2) this can be written as

$$\frac{\partial \tilde{L}}{\partial \dot{z}} = \frac{\partial L}{\partial \dot{y}}\frac{\partial y}{\partial z}$$

Then,

$$\frac{d}{dx}\frac{\partial \tilde{L}}{\partial \dot{z}} = \left(\frac{d}{dx}\frac{\partial L}{\partial \dot{y}}\right)\frac{\partial y}{\partial z} + \frac{\partial L}{\partial \dot{y}}\left(\frac{d}{dx}\frac{\partial y}{\partial z}\right)$$

Now we can write down the E-L equation in the new coordinates,

$$\frac{d}{dx}\frac{\partial \tilde{L}}{\partial \dot{z}} - \frac{\partial \tilde{L}}{\partial z} = \left(\frac{d}{dx}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y}\right)\frac{\partial y}{\partial z}$$

So as long as $\frac{\partial y}{\partial z} \neq 0$ (i.e. the coordinate transformation is well-defined), the E-L equation in the new coordinates is equivalent to the old one.

Generalization to multiple coordinates $\vec{y}$ and $\vec{z}$ is straightforward.

1.5 Lagrangians

The functional that governs the motion is the **action**.

$$\boxed{S[\vec{r}(t)] = \int_{t_1}^{t_2} dt\, L(\vec{r}(t),\, \dot{\vec{r}}(t),\, t)}$$

where

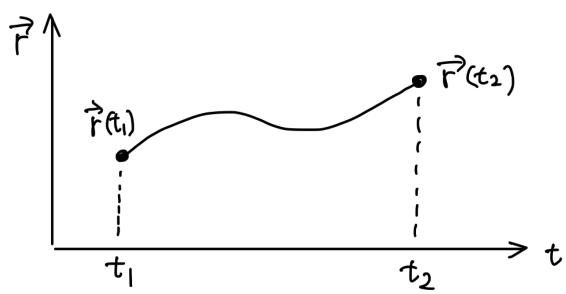
$$\boxed{L = T - V = \frac{1}{2}m\dot{\vec{r}}^2 - V(\vec{r},t) \qquad \text{Lagrangian}}$$

where $\vec{r}$ denotes Cartesian coordinates.

- Note that L is **not** the energy.

1.6 Hamilton's principle of least action

$$\boxed{\text{The motion of a system is the one that extremizes the action } \; S[\vec{r}(t)]}$$



Proof:

$\vec{r}(t)$ extremizes $S[\vec{r}(t)]$ means that it satisfies the E-L equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}} - \frac{\partial L}{\partial \vec{r}} = 0$$

where $L = \frac{1}{2}m\dot{\vec{r}}^2 - V(\vec{r},t)$.

Now

$$\frac{\partial L}{\partial \dot{\vec{r}}} = m\dot{\vec{r}} \equiv \vec{p} \qquad \text{momentum conjugate to } \vec{r}$$

Hence

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}} = \dot{\vec{p}}$$

The other term in the E-L equation:

$$\frac{\partial L}{\partial \vec{r}} = -\frac{\partial V}{\partial \vec{r}} = -\vec{\nabla} V$$

So we have

$$\dot{\vec{p}} = -\vec{\nabla} V$$

which is Newton's equation of motion. **Q.E.D.**

• The principle gives a useful vantage point for the theory of mechanics. It is regarded as *more fundamental* than Newton's 2nd law.

• Hamilton's principle of least action suggests that Nature "tries" all possible virtual paths and singles out the one that extremizes the action. (This is made precise in Feynman's path integral formulation of quantum mechanics.)

1.7 Lagrangians differing by a total derivative describe the same physics

Let's suppose $\tilde{L} = L + \frac{d}{dt}F(q,t)$ (note: no $\dot{q}$ in F).

Does it give the same E-L equation? Yes!

Proof:

$$S[q] \equiv \int_{t_1}^{t_2} dt\, L$$

$$\tilde{S}[q] \equiv \int_{t_1}^{t_2} dt\, \tilde{L} = S + \int_{t_1}^{t_2} dt \frac{d}{dt}F = S + F(q(t_2), t_2) - F(q(t_1), t_1)$$

But variations are taken with fixed endpoints, namely $q(t_1) = q_1$ and $q(t_2) = q_2$ are fixed. Thus, the E-L equations are the same for L and $\tilde{L}$. **Q.E.D.**

1.8 Example: A particle in Cartesian coordinates

In the free case $V = 0$ and the Lagrangian is

$$L = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

The Euler-Lagrange equations are

$$\begin{cases} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0 \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = 0 \end{cases}$$

More explicitly,

$$\begin{cases} \frac{d}{dt}(m\dot{x}) = 0 \\ \frac{d}{dt}(m\dot{y}) = 0 \\ \frac{d}{dt}(m\dot{z}) = 0 \end{cases}$$

which implies that $\vec{p} = const.$

If we turn on gravity, $V = mgz$, and $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$.

The E-L equations for the x and y coordinates are unchanged. For z,

$$\frac{\partial L}{\partial \dot{z}} = m\dot{z}, \qquad \frac{\partial L}{\partial z} = -mg$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = \frac{\partial L}{\partial z} \qquad \Rightarrow \qquad m\ddot{z} = -mg$$

1.9 Example: N free particles in Cartesian coordinates

$$L = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\vec{r}}_i^2$$

$$\vec{p}_i = \frac{\partial L}{\partial \dot{\vec{r}}_i} = m\dot{\vec{r}}_i, \qquad \dot{\vec{p}}_i = 0$$

1.10 Generalized coordinates

A set of coordinates $(q_1, \ldots, q_n) \equiv \vec{q}$ that completely defines the positions of the system.

$$\#DoF = n \leq 3N \quad (N \text{ is the number of particles})$$

The action S depends on the trajectory (the path of motion) itself and not on the coordinates. Thus, we can use any set of generalized coordinates $q_i(t)$ to write down the Euler-Lagrange equations, and find the motion via solving

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}}$$

• Actually, there is a gap in the logic: we have shown the equivalence between the E-L equations and Newton's 2nd law only for the case $n = 3N$, namely in the absence of *constraints* that reduce the number of DoF.

• As it turns out, the E-L equations are correct even for systems with $n < 3N$ (for systems with constraints).

Some definitions:

• The quantities $p_{q_i} \equiv \frac{\partial L}{\partial \dot{q}_i}$ are the **generalized momenta**.

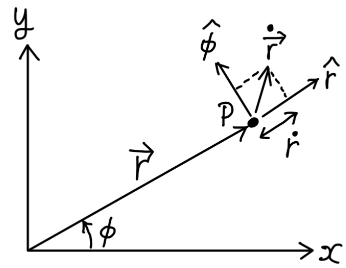
• The quantities $Q_i \equiv \frac{\partial L}{\partial q_i}$ are the **generalized forces**.

In terms of these, the Euler-Lagrange equation can be written as

$$\dot{p}_{q_i} = Q_i$$

• If the Lagrangian does not contain for instance q_i for some i, then the $Q_i = 0$. In this case q_i is said to be **cyclic** and the corresponding momentum p_{q_i} is conserved.

1.11 Example: Particle moving in a plane (polar coordinates)



$$\vec{r} = r\hat{r}$$

$$\dot{\vec{r}} = \frac{d}{dt}(r\hat{r}) = \dot{r}\hat{r} + r\dot{\hat{r}}$$

Clearly, $\dot{\hat{r}} = \dot{\phi}\hat{\phi}$ where $|\hat{\phi}| = 1$, and $\hat{\phi}\cdot\hat{r} = 0$.

So

$$\boxed{\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}}$$

This the kinetic energy,

$$T = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}m(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi})^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$$

which is independent of ϕ.

$$\boxed{T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)}$$

Alternatively,

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases} \Rightarrow \begin{cases} \dot{x} = \dot{r}\cos\phi - r(\sin\phi)\dot{\phi} \\ \dot{y} = \dot{r}\sin\phi + r(\cos\phi)\dot{\phi} \end{cases}$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$$

• Generalized momenta:

$$p_r = \frac{\partial L}{\partial \dot{r}}, \qquad p_\phi = \frac{\partial L}{\partial \dot{\phi}}$$

If

$$L = T(r, \phi, \dot{r}) - V(\vec{r})$$

where V is independent of $\dot{r}$ and $\dot{\phi}$, then

$$\boxed{p_r = m\dot{r}, \qquad p_\phi = mr^2\dot{\phi}}$$

and the Euler-Lagrange equations are

$$\dot{p}_r = \underbrace{mr\dot{\phi}^2}_{\text{centrifugal force}} - \frac{\partial V}{\partial r}, \qquad \dot{p}_\phi = -\frac{\partial V}{\partial \phi}$$

If we further have $V = V(r)$ (i.e. the potential is independent of ϕ), then ϕ is a cyclic coordinate and p_ϕ =const (conserved). In this case the force is central and angular momentum is conserved.