SPA5304 Physical Dynamics Lecture 7

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1 Many particles

1.1 Conservative forces

1.1.1 External forces

Assume external forces are conservative:

$$\vec{F}_i^{(e)} = -\vec{\nabla}_i V_i(\vec{r}_i)$$

where $\vec{\nabla}_i$ is the gradient with respect to $\vec{r}_i$.

The work done by external forces:

$$W_{\text{ext}} = \sum_i \int_1^2 \vec{F}_i^{(e)} \cdot d\vec{r}_i = -\sum_i \int_1^2 d\vec{r}_i \cdot \vec{\nabla}_i V_i = -\sum_i V_i|_1^2 = \sum_i (V_i(1) - V_i(2))$$

1.1.2 Internal forces

Assuming the strong law of action and reaction, the force on particle i exerted by particle j can be written as:

$$\vec{F}_{ij} = \hat{r}_{ij} f_{ij} \qquad \hat{r}_{ij} = \frac{\vec{r}_{ij}}{|\vec{r}_{ij}|} \qquad \vec{r}_{ij} = \vec{r}_i - \vec{r}_j$$

and $f_{ij} = f_{ji}$ is some scalar function.

- Now if f_{ij} depends only on the relative distance $|\vec{r}_{ij}|$ (we will assume this henceforth), then $\vec{F}_{ij}$ is automatically conservative.

Proof:

Define the potential

$$V_{ji} \equiv \int_0^{|\vec{r}_{ji}|} d\rho \, f_{ji}(\rho)$$

Then by the chain rule

$$-\vec{\nabla}_i V_{ji} = -f_{ji}(|\vec{r}_{ji}|)\vec{\nabla}_i |\vec{r}_{ji}|$$

Here

$$\vec{\nabla}_i |\vec{r}_{ji}| = \vec{\nabla}_i \sqrt{(\vec{r}_j - \vec{r}_i) \cdot (\vec{r}_j - \vec{r}_i)} = -\frac{\vec{r}_j - \vec{r}_i}{\sqrt{(\vec{r}_j - \vec{r}_i) \cdot (\vec{r}_j - \vec{r}_i)}} = -\hat{r}_{ji}$$

This means that

$$-\vec{\nabla}_i V_{ji} = +\hat{r}_{ji} f_{ji}(|\vec{r}_{ji}|) = \vec{F}_{ji}$$

Q.E.D.

Remark: We have $V_{ij} = V_{ji}$. Then the stong law $\vec{F}_{ij} = -\vec{F}_{ji}$ comes from antisymmetry of

$$\vec{\nabla}_i |\vec{r}_{ij}| = \vec{\nabla}_i |\vec{r}_i - \vec{r}_j| = -\vec{\nabla}_j |\vec{r}_i - \vec{r}_j| = -\vec{\nabla}_j |\vec{r}_{ij}|$$

Let us now compute the work done by internal forces.

$$W_{\text{int}} = \sum_i d\vec{r}_i \cdot \vec{F}_i^{(\text{int})} = \sum_{i,j} \int d\vec{r}_i \cdot \vec{F}_{ji} = -\sum_{i,j} \int d\vec{r}_i \cdot \vec{\nabla}_i V_{ji}$$

Use $V_{ij} = V_{ji}$ and relabel dummy variables ($i \leftrightarrow j$) to get

$$W_{\text{int}} = -\frac{1}{2} \sum_{i,j} \int d\vec{r}_i \cdot \vec{\nabla}_i V_{ji} - \frac{1}{2} \sum_{i,j} \int d\vec{r}_j \cdot \underbrace{\vec{\nabla}_j V_{ji}}_{-\vec{\nabla}_i V_{ji}} = -\frac{1}{2} \sum_{i,j} \int (d\vec{r}_i - d\vec{r}_j) \cdot \vec{\nabla}_i V_{ji}$$

Now we can define

$$\vec{\nabla}_i V_{ji}(|\vec{r}_{ij}|) = \frac{\partial}{\partial \vec{r}_i} V_{ji}(|\vec{r}_{ij}|) = \frac{\partial}{\partial \vec{r}_{ij}} V_{ji}(|\vec{r}_{ij}|) \equiv \vec{\nabla}_{ij} V_{ji}$$

to write

$$W_{\text{int}} = -\frac{1}{2} \sum_{i,j} \int d\vec{r}_{ij} \cdot \vec{\nabla}_{ij} V_{ji} = -\frac{1}{2} \left[\sum_{i,j} V_{ij} \right]_1^2$$

Including the external part, the total work is

$$\boxed{W = W_{\text{ext}} + W_{\text{int}} = -\left[\sum_i V_i + \frac{1}{2} \sum_{i,j} V_{ij} \right]_1^2}$$

This is independent of the path (as in the single-particle case).

Recall that $W = [\sum_i T_i]_1^2$. Hence, the total energy E is conserved:

$$\boxed{E = \sum_i T_i + \underbrace{\sum_i V_i + \frac{1}{2} \sum_{i,j} V_{ij}}_{\textbf{total potential energy}}}$$

The last term (the **internal potential energy**) can also be written as $\frac{1}{2} \sum_{i,j} V_{ij} = \sum_{i<j} V_{ij}$.

1.2 Rigid bodies

A rigid body is a system of particles for which $|\vec{r}_{ij}| = \text{const.}$

- For a rigid body, the internal potential energy is constant in time.

Hence, no work is done by internal forces.

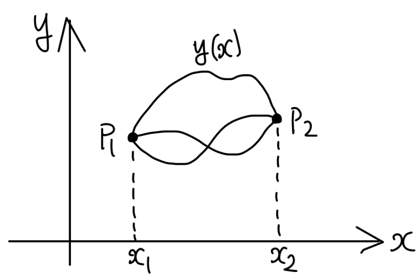
- A rigid body is an example for a **holonomic constraint**. Such constraints are of the form

$$f_K(\vec{r}_1, \ldots, \vec{r}_N, t) = 0$$

2 Lagrangian mechanics

2.1 Calculus of variations

A **functional** is a function of a function.



Given a function $y(x)$, a number is determined: $F[y]$.

2.1.1 Example: length of a curve

In this example, the functional is the length of a curve whose graph is given by $y = y(x)$.

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \dot{y}^2} \qquad \text{where} \quad \dot{y} \equiv \frac{dy}{dx}$$

- The $l[y]$ length of the curve between points P_1 and P_2 is

$$l[y] = \int_{x_1}^{x_2} dx\sqrt{1 + \dot{y}^2}$$

- It is clear that $l[y]$ is minimized when the path is a straight line. But how does one prove this?

We will be interested in the minimum/maximum, i.e. the **extremum** of functionals.

For a function $f(x)$, we know that $\frac{df}{dx} = 0$ gives an extremum:

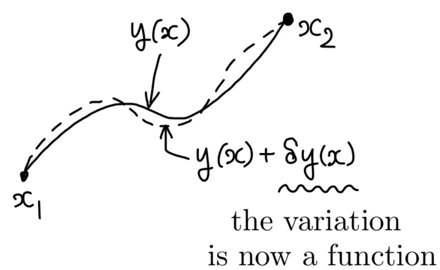
$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = 0 \qquad \Delta x : \text{small “variation”}$$

or we can simply write

$$\delta f \equiv f(x+\Delta x) - f(x) = 0$$

Namely, we vary x by a small (infinitesimal) amount, Δx. If the change in $f(x)$ vanishes at the linear level, then $f(x)$ is extremized (it is “stationary”).

• The situation is simlar for a functional $F[y]$. Vary the function $y(x)$ by a small (infinitesimal) amount as in the figure:



The change in F must vanish for an extremum:

$$\boxed{\delta F \equiv F[y+\delta y] - F[y] = 0}$$

(here we ignore higher order, i.e. $\mathcal{O}(\delta y^2)$ terms).

In general, we will consider functionals which depend on y, $\dot{y}$, and x. They take the form

$$\boxed{F[y] = \int_{x_1}^{x_2} dx\, L\left(y(x),\, \dot{y}(x),\, x\right)}$$

Our previous example $l[y]$ had this form (L did not explicitly depend on $y(x)$).