SPA5304 Physical Dynamics Lecture 5-6

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1 Many particles

[Ref: Goldstein §1.2]

Consider multiple particles labeled by $i = 1, 2, \ldots$

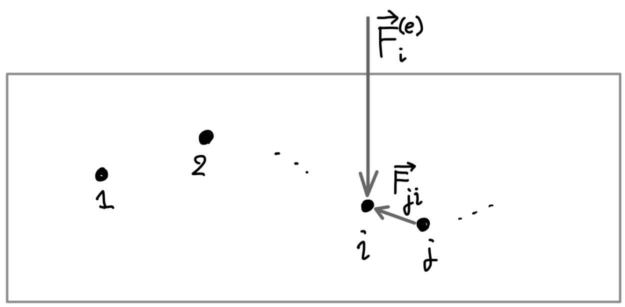
1.1 Internal vs. external forces

- Particle i may feel an **external force** $\vec{F}_i^{(e)}$

  e.g. External electric field, when the particles are charged

- Particle i may feel an **internal force** exerted by particle j $\vec{F}_{ji}$

  e.g. Coulomb force between particle i and j.



1.2 Newton's laws

Newton's 2nd law for particle i:

$$\dot{\vec{p}}_i = \vec{F}_i^{(e)} + \sum_{j \neq i}^{N} \vec{F}_{ji}$$

Here the sum excludes $j = i$. But we can include $j = i$ if we define $\vec{F}_{ji} = 0$. This will be understood henceforth.

• Newton's 3rd law:

$$\boxed{\vec{F}_{ji} = -\vec{F}_{ij}}$$

1.3 Total momentum

$$\vec{p} \equiv \sum_i \vec{p}_i = \sum_i m_i \vec{v}_i$$

Sum the 2nd law equation over i:

$$\dot{\vec{P}} = \sum_i \dot{\vec{p}}_i = \sum_i \vec{F}_i^{(e)} + \underbrace{\sum_{i,j} \vec{F}_{ji}}_{=0 \text{ by 3rd law}}$$

Thus,

$$\dot{\vec{P}} = \sum_i \vec{F}_i^{(e)} = \text{total external force} \equiv \vec{F}$$

1.4 Center of mass

$$\boxed{\vec{R} \equiv \frac{\sum_i m_i \vec{r}_i}{M}}$$

where

$$M \equiv \sum_i m_i$$

is the total mass.

$\vec{R}$ is the average position weighted by mass.

$$M\vec{R} = \sum_i m_i \vec{r}_i$$

$$M\dot{\vec{R}} = \sum_i m_i \dot{\vec{r}}_i = \sum_i \vec{p}_i = \vec{P} \qquad \text{total momentum}$$

$$M\ddot{\vec{R}} = \sum_i m_i \ddot{\vec{r}}_i = \sum_i \vec{F}_i^{(e)} = \dot{\vec{P}} = \vec{F} \qquad \text{total external force}$$

Hence,

$$\boxed{\vec{P} = M\dot{\vec{R}}}$$

and

$$\boxed{\dot{\vec{P}} = M\ddot{\vec{R}} = \vec{F}}$$

- The COM (center of mass) moves as if it were a particle with mass M, acted on by a force $\vec{F}$.
- If the total force vanishes ($\vec{F} = 0$), then the total momentum is conserved: $\vec{P} = \text{const.}$

1.5 Total angular momentum

Angular momentum of particle i about $\vec{r} = 0$: $\quad \vec{L}_i = \vec{r}_i \times \vec{p}_i$

Total angular momentum about $\vec{r} = 0$:

$$\boxed{\vec{L} \equiv \sum_i \vec{L}_i = \sum_i \vec{r}_i \times \vec{p}_i}$$

What is the time evolution of $\vec{L}$?

$$\dot{\vec{L}}_i = \underbrace{\dot{\vec{r}}_i \times \vec{p}_i}_{=0} + \vec{r}_i \times \dot{\vec{p}}_i = \vec{r}_i \times \left(\vec{F}_i^{(e)} + \sum_j \vec{F}_{ji} \right)$$

Let's sum over i:

$$\dot{\vec{L}} = \sum_i \dot{\vec{L}}_i = \sum_i \vec{r}_i \times \vec{F}_i^{(e)} + \sum_{i,j} \vec{r}_i \times \vec{F}_{ji}$$

Since i and j are dummy variables, we can swap them and write the second term as

$$\sum_{i,j} \vec{r}_i \times \vec{F}_{ji} = \frac{1}{2} \sum_{i,j} \vec{r}_i \times \vec{F}_{ji} + \frac{1}{2} \sum_{i,j} \vec{r}_j \times \underbrace{\vec{F}_{ij}}_{\substack{-\vec{F}_{ji} \\ \text{by } 3^{rd} \text{ law}}} = \frac{1}{2} \sum_{i,j} (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji}$$

Assuming the "strong law of action and reaction" we can write the internal force as

$$\vec{F}_{ij} = \hat{r}_{ij} f_{ij} \qquad \hat{r}_{ij} = \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|}$$

and $f_{ij} = f_{ji}$ is some scalar function. This means that the force vector lies along the straight line joining particles i and j.

Hence $(\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji} = 0$ and thus

$$\dot{\vec{L}} = \sum_i \dot{\vec{L}}_i = \underbrace{\sum_i \vec{r}_i \times \vec{F}_i^{(e)}}_{\text{total external torque}} \tag{1}$$

1.6 Role of COM in total angular momentum

The time evolution of the total linear momentum $\vec{P}$ had a nice interpretation in terms of the center of mass vector $\vec{R}$ (namely $\vec{P} = M\dot{\vec{R}}$). What about the total angular momentum $\vec{L}$?

Let's decompose $\vec{r}_i$ as

$$\vec{r}_i = \vec{R} + \vec{r}_i{}'$$

where $\vec{r}_i{}'$ is the relative position w.r.t. the center of mass.

Take the time-derivative:

$$\dot{\vec{r}}_i = \dot{\vec{R}} + \dot{\vec{r}}_i{}'$$

and multiply by m_i

$$\underbrace{m_i \dot{\vec{r}}_i}_{=\vec{p}_i} = m_i \dot{\vec{R}} + \underbrace{m_i \dot{\vec{r}}_i{}'}_{\substack{\equiv \vec{p}_i{}' \\ \text{by definition}}}$$

Then,

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i (\vec{R} + \vec{r}_i{}') \times \left(m_i \dot{\vec{R}} + \vec{p}_i{}' \right) =$$

$$= \underbrace{\vec{R} \times \left(\sum_i m_i \right) \dot{\vec{R}}}_{=\vec{P}} + \vec{R} \times \sum_i \vec{p}_i{}' + \left(\sum_i m_i \vec{r}_i{}' \right) \times \dot{\vec{R}} + \sum_i \vec{r}_i{}' \times \vec{p}_i{}' \tag{2}$$

Here

$$\sum_i m_i \vec{r}_i{}' = \sum_i m_i (\vec{r}_i - \vec{R}) = \underbrace{\sum_i m_i \vec{r}_i}_{=M\vec{R}} - \left(\sum_i m_i \right) \vec{R} = 0$$

If we take the time-derivative, we also get

$$\sum_i \vec{p}_i{}' = 0$$

Thus, two terms vanish in (2) and we get

$$\boxed{\vec{L} = \vec{R} \times \vec{P} + \sum_i \vec{r}_i{}' \times \vec{p}_i{}'}$$

- The first term is the "orbital" part and it comes from the COM motion.
- The second term will be denoted by $\vec{L}' \equiv \sum_i \vec{r}_i{}' \times \vec{p}_i{}'$. It is the "spin" part of the angular momentum.

(E.g. in the case of the Earth, the orbital part is the contribution to angular momentum from Earth's revolution around the Sun. The spin part is the contribution from Earth's rotation.)

1.7 Time evolution of $\vec{L}'$

$$\vec{L} = \vec{R} \times \vec{P} + \vec{L}'$$

$$\dot{\vec{L}} = \underbrace{\dot{\vec{R}} \times \vec{P}}_{=0} + \vec{R} \times \underbrace{\dot{\vec{P}}}_{=\sum_i \vec{F}_i^{(e)}} + \dot{\vec{L}}'$$

On the other hand from eqn. (1) we have

$$\dot{\vec{L}} = \sum \vec{r}_i \times \vec{F}_i^{(e)}$$

Comparing the two expressions for $\dot{\vec{L}}$, we get

$$\dot{\vec{L}}' = \sum \vec{r}_i \times \vec{F}_i^{(e)} - \vec{R} \times \vec{F}_i^{(e)}$$

$$\boxed{\dot{\vec{L}}' = \sum \vec{r}_i{}' \times \vec{F}_i^{(e)}}$$

On the RHS is the sum of external torques about the COM.

1.8 Work done by forces in a many-body system

A "configuration" is a set of values for the position vectors $\{\vec{r}_i\}$.

Let us now move the system from **configuration 1** to **configuration 2**. The work done in the process:

$$W[\mathcal{P}] = \sum_i \int_1^2 d\vec{r}_i \cdot \vec{F}_i = \sum_i \int_1^2 d\vec{r}_i \cdot \left(\vec{F}_i^{(e)} + \sum_j \vec{F}_{ji} \right)$$

$$= \sum_i \int_{t_1}^{t_2} dt \frac{d\vec{r}_i}{dt} \cdot \vec{F}_i = \sum_i \int_{t_1}^{t_2} dt \frac{\vec{p}_i}{m_i} \cdot \dot{\vec{p}}_i = \int_{t_1}^{t_2} dt \frac{d}{dt} \left(\sum_i \frac{\vec{p}_i{}^2}{2m_i} \right)$$

$$= \sum_i (T_i(2) - T_i(1)) = T_2 - T_1$$

where $T_i \equiv \frac{\vec{p}_i{}^2}{2m_i}$ is the kinetic energy for particle i.

The result above is the same as in the 1-particle case: the (total) kinetic energy of the system changes by the work done to the system.

1.9 Role of the COM in the kinetic energy

Let us decompose T into the COM and relative parts

$$T = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} = \sum_i \frac{1}{2} m_i \left(\dot{\vec{R}} + \dot{\vec{r}}_i{}'\right)^2 = \sum_i \frac{1}{2} m_i \left[\left(\dot{\vec{R}}\right)^2 + (\dot{\vec{r}}_i{}')^2 + 2\dot{\vec{R}} \cdot \dot{\vec{r}}_i{}'\right]$$

$$= \frac{1}{2} M \left(\dot{\vec{R}}\right)^2 + \frac{1}{2} \sum_i m_i (\dot{\vec{r}}_i{}')^2 + \dot{\vec{R}} \cdot \frac{d}{dt} \underbrace{\sum_i m_i \vec{r}_i{}'}_{=0}$$

$$\boxed{T = \frac{1}{2} M \left(\dot{\vec{R}}\right)^2 + \frac{1}{2} \sum_i m_i (\dot{\vec{r}}_i{}')^2 = T_{\text{COM}} + T'}$$

(Note that this decomposition is similar to that of the angular momentum $\vec{L} = \vec{R} \times \vec{P} + \vec{L}'$)

1.10 Summary

$$\dot{\vec{p}}_i = \underbrace{\vec{F}_i^{(e)}}_{\substack{\text{external}\\\text{forces}}} + \underbrace{\sum_j \vec{F}_{ji}}_{\substack{\text{internal}\\\text{forces}}}$$

Decomposition: $\vec{r}_i = \vec{R} + \vec{r}_i{}'$ center of mass: $\vec{R} \equiv \frac{\sum_i m_i \vec{r}_i}{M}$ total mass: $M \equiv \sum_i m_i$.

	definition	decomposition	time evolution
total linear momentum	$\vec{P} = \sum_i \vec{p}_i$	$\vec{P} = M\dot{\vec{R}}$	$\dot{\vec{P}} = \vec{F} = \sum_i \vec{F}_i^{(e)}$
total angular momentum	$\vec{L} = \sum_i \vec{L}_i$	$\vec{L} = \underbrace{\vec{R} \times \vec{P}}_{\text{"orbital"}} + \underbrace{\sum_i \vec{r}_i{}' \times \vec{p}_i{}'}_{\text{"spin"}}$	$\dot{\vec{L}} = \sum \vec{r}_i \times \vec{F}_i^{(e)}$ $\dot{\vec{L}}' = \sum \vec{r}_i{}' \times \vec{F}_i^{(e)}$
total kinetic energy	$T = \sum_i T_i$	$T = \frac{1}{2} M \left(\dot{\vec{R}}\right)^2 + \frac{1}{2} \sum_i m_i (\dot{\vec{r}}_i{}')^2$	