SPA5304 Physical Dynamics Lecture 4

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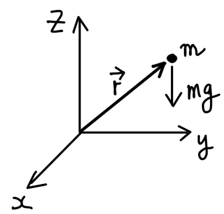
1 Examples of conservative systems

1.1 Particle in a gravitational field



$$V = +mgz$$

$$T = \frac{1}{2}m\vec{v}^2 = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

- Total energy:

$$E = T + V = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + mgz \tag{1}$$

- Equation of motion (Newton's 2nd law);

$$\dot{\vec{p}} = \vec{F} = -\frac{\partial V}{\partial \vec{r}} = \begin{cases} m\ddot{x} = 0 \\ m\ddot{y} = 0 \\ m\ddot{z} = -mg \end{cases}$$

From which we have

$$\begin{cases} \ddot{x} = 0 \\ \ddot{y} = 0 \\ \ddot{z} = -g \end{cases} \tag{2}$$

These can be integrated. Then we get

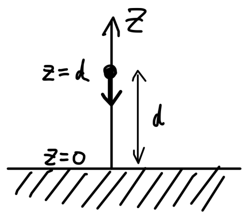
$$\begin{cases} \dot{x}(t) = \text{const} = \dot{x}(0) \\ \dot{y}(t) = \text{const} = \dot{y}(0) \\ \dot{z}(t) = -gt + \text{const} = -gt + \dot{z}(0) \end{cases} \tag{3}$$

Integrate again:

$$\begin{cases} x(t) = \dot{x}(0)t + x(0) \\ y(t) = \dot{y}(0)t + y(0) \\ z(t) = -\frac{1}{2}gt^2 + \dot{z}(0)t + z(0) \end{cases} \tag{4}$$

As a check, one can plug these results back into the total energy in (1). It is easy to show that E is independent of t, i.e. energy is conserved.

1.2 Application of energy conservation: Freely falling particle



- Initially (at $t = 0$), the particle is at rest.
- At time t_* the particle hits the ground with velocity v_*.

Let us determine v_*. Compute the initial and final total energies:

$$E(t = 0) = T(t = 0) + V(t = 0) = 0 + mgd$$

$$E(t_*) = T(t_*) + V(t_*) = \frac{1}{2}mv_*^2 + 0$$

Using energy conservation, i.e. $E(0) = E(t_*)$ we get

$$mgd = \frac{1}{2}mv_*^2$$

which gives

$$v_* = \sqrt{2gd}$$

To find t_*, energy conservation is not enough. Using the previous result,

$$z(t_*) = -\frac{1}{2}gt_*^2 + \underbrace{\dot{z}(0)}_{=0}t_* + \underbrace{z(0)}_{=d}$$

$$0 = -\frac{1}{2}gt_*^2 + d$$

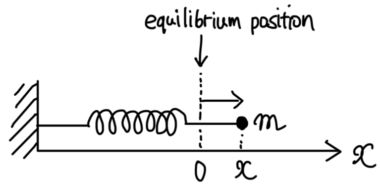
from which we get

$$t_* = \sqrt{\frac{2d}{g}}$$

This formula is independent of the particle mass m (as Galilei noted).

In some systems, we can also use conservation of (angular) momentum to solve problems (see Exercise Class and Homework).

1.3 Another example for a conservative system: Harmonic oscillator



$$F = -kx, \quad V = \frac{1}{2}kx^2$$

$$E = T + V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

The equation of motion is:

$$m\ddot{x} = -\frac{dV}{dx} = -kx$$

$$\ddot{x} = -\frac{k}{m}x \equiv -\omega^2 x, \qquad \omega \equiv \sqrt{\frac{k}{m}}$$

What is this ω frequency? Let's see the solution:

$$x(t) = A\cos\omega t + B\sin\omega t$$

This gives for the initial position and velocity:

$$x(0) = A$$

$$\dot{x}(0) = B\omega$$

We can express A and B and get,

$$x(t) = x(0)\cos\omega t + \frac{\dot{x}(0)}{\omega}\sin\omega t$$

Let us calculate the energy as well. For this we will need the velocity which we can calculate from $x(t)$:

$$\dot{x}(t) = -\omega x(0) \sin \omega t + \dot{x}(0) \cos \omega t$$

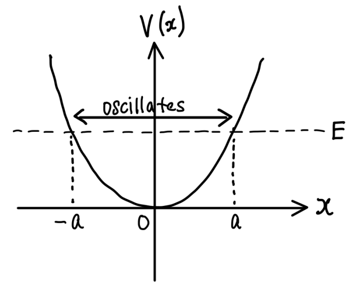
$$E(t) = \frac{1}{2} m \left[-\omega x(0) \sin \omega t + \dot{x}(0) \cos \omega t\right]^2 + \frac{1}{2} k \left[x(0) \cos \omega t + \frac{\dot{x}(0)}{\omega} \sin \omega t\right]^2$$

$$= \frac{1}{2} x(0)^2 \left(m\omega^2 \sin^2 \omega t + k \cos^2 \omega t\right) + \frac{1}{2} \dot{x}(0)^2 \left(m \cos^2 \omega t + \frac{k}{\omega^2} \sin^2 \omega t\right) + x(0)\dot{x}(0) \left(-m\omega + \frac{k}{\omega}\right) \cos \omega t \sin \omega t$$

$$E(t) = \frac{k}{2} x(0)^2 + \frac{m}{2} \dot{x}(0)^2 = \frac{k}{2}(A^2 + B^2) = \text{const}$$

Independent of t, i.e. energy is conserved!

1.3.1 The quadratic potential



$$V = \frac{1}{2} k x^2$$

$$x(t) = A \cos \omega t + B \sin \omega t = a \cos(\omega t + \alpha)$$

$$E = \frac{k}{2}(A^2 + B^2) = \frac{k}{2} a^2$$

where $a \equiv \sqrt{A^2 + B^2}$.

The system oscillates back and forth over the values of x for which $E - V(x) \geq 0$. This observation generalizes to other cases.

1.4 A qualitative study of motion from the potential (in one dimension)

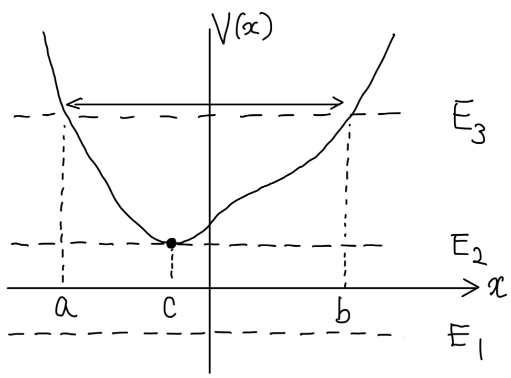
$$E = \frac{1}{2}m\vec{v}^2 + V(\vec{r})$$

$$\frac{1}{2}m\vec{v}^2 = E - V \geq 0$$

Note: $E - V = 0$ if and only if $\vec{v} = 0$.

By looking at the shape of the potential, we can qualitatively tell how a particle moves.

Consider the one-dimensional case:



Let's look at different cases:

$E = E_3$	$E_3 - V \geq 0$ only for $a \leq x \leq b$. Finite oscillatory motion between $x = a$ and $x = b$. These two points are called **turning points**.
$E = E_2$	$E_2 - V \geq 0$ only at $x = c$ where $v = 0$ (static system). $x = c$ is called an **equilibrium position**. Note: $V'(c) = 0$.
$E = E_1$	$E_1 < V$ for all x. No physical motion is possible.

1.5 Stable and unstable equilibrium points

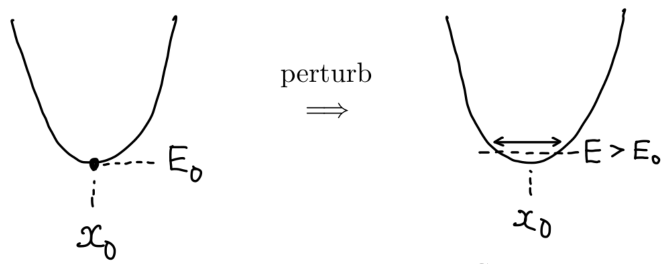
- Equilibrium position $\Leftrightarrow$ $V'(x) = 0$ for some x.

If $V(x)$ has a (local) minimum there ($V''(x) > 0$) $\Rightarrow$ **Stable equilibrium**

If $V(x)$ has a (local) maximum there ($V''(x) < 0$) $\Rightarrow$ **Unstable equilibrium**

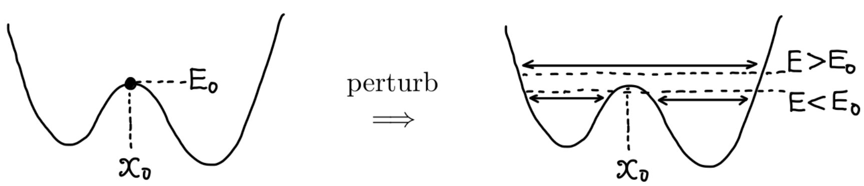
- The stability is related to what happens if one perturbs the system

Stable



Stays near equilibrium position

Unstable



Does *not* stay near equilibrium position