SPA5304 Physical Dynamics Lecture 2-3

David Vegh

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0.1 Newton's Laws of Motion

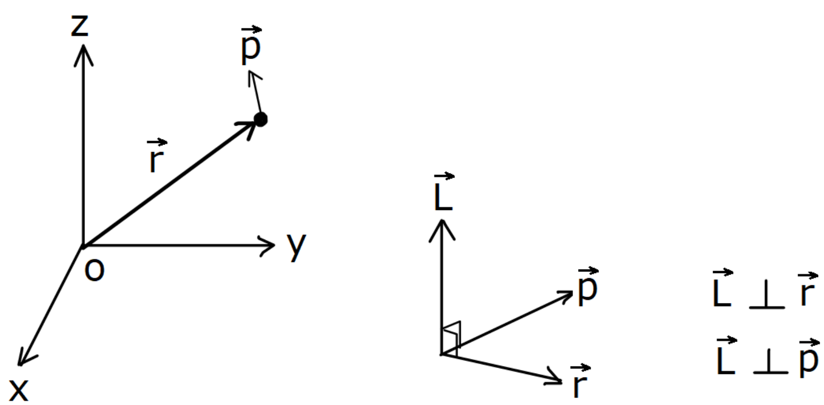
1^{st} law: Objects remain at rest or move at constant velocity unless acted upon by a force (inertia)

2^{nd} law: $\dot{\vec{p}} = \vec{F}$

3^{rd} law: $\vec{F}_{12} = -\vec{F}_{21}$ (action-reaction)

0.2 Angular momentum

The **angular momentum** of a particle with $\vec{p}$ about point "0"



$$\boxed{\vec{L} \equiv \vec{r} \times \vec{p}}$$

This $\vec{L}$ is about 0. The angular momentum about some other $\vec{r}_0$ is

$$\vec{L} \equiv (\vec{r} - \vec{r}_0) \times \vec{p}$$

• Components:

$$\vec{r}\times\vec{p} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = \hat{x}(yp_z - zp_y) + \hat{y}(zp_x - xp_z) + \hat{z}(xp_y - yp_x)$$

Thus,

$$\vec{L} = (yp_z - zp_y,\, zp_x - xp_z,\, xp_y - yp_x)$$

Or, if we use (x_1, x_2, x_3) and (p_1, p_2, p_3) instead,

$$L_1 = x_2p_3 - x_3p_2$$

$$L_2 = x_3p_1 - x_1p_3$$

$$L_3 = x_1p_2 - x_2p_1$$

The relations are cyclic.

• Note: $\vec{p} = m\vec{v}$ is sometimes called **linear momentum**.

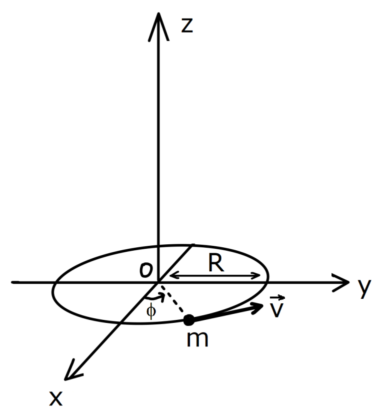
0.2.1 Example: Constant revolution

$$\vec{r} = R(\cos\phi,\, \sin\phi,\, 0)$$

$$\vec{v} = v(-\sin\phi,\, \cos\phi,\, 0)$$

$$\vec{p} = mv(-\sin\phi,\, \cos\phi,\, 0)$$

$$\vec{L} = \vec{r}\times\vec{p} = Rmv\left(0,\, 0,\, \cos^2\phi + \sin^2\phi\right) = (0, 0, Rmv)$$



• More compactly,

$$L_i = \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} x_j p_k \qquad \text{where} \qquad i = 1, 2, 3$$

Here ε_{ijk} is the **Levi-Civita symbol**, or "ε-symbol". It is totally antisymmetric, and $\varepsilon_{123} = +1$.

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1$$

$$\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$$

while the other components are zero.

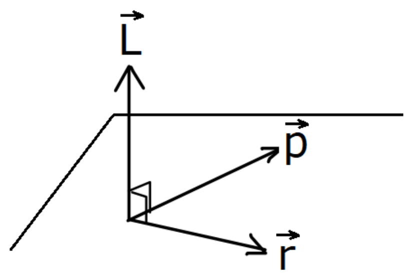
• Using the rule that repeated indices imply summation:

$$(\vec{A} \times \vec{B})_i = \varepsilon_{ijk} A_j B_k$$

$$\boxed{L_i = \varepsilon_{ijk} r_j p_k}$$

• One can show that $\vec{r} \times \vec{p} = -\vec{p} \times \vec{r}$.

$$\vec{L} \cdot \vec{r} = \vec{L} \cdot \vec{p} = 0 \quad \Rightarrow \quad \vec{L} \perp \vec{r}, \; \vec{L} \perp \vec{p}$$



$\vec{L}$ is orthogonal to the plane spanned by $\vec{r}$ and $\vec{p}$.

0.3 Time evolution of $\vec{L}$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \dot{\vec{r}} \times \underbrace{\vec{p}}_{m\dot{\vec{r}}} + \vec{r} \times \underbrace{\dot{\vec{p}}}_{\vec{F}}$$

Since $\dot{\vec{r}} \times \dot{\vec{r}} = 0$, we get

$$\boxed{\dot{\vec{L}} = \vec{r} \times \vec{F} \equiv \vec{\tau}}$$

$\vec{\tau}$ is the moment of force about "0", or **torque**.

0.4 Conservation of angular momentum

If torque is zero, $\vec{r} \times \vec{F} = 0$ (here $\vec{F}$ is the total force) $\Rightarrow \quad \dot{\vec{L}} = 0$, which means

$$\boxed{\vec{L} = \text{const.}}$$

i.e. angular momentum is conserved.

0.4.1 Example: Central force

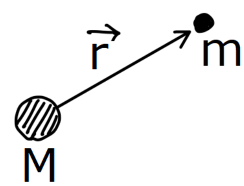
• Let the force be

$$\vec{F}(\vec{r}) = f(r)\hat{r}, \qquad r = |\vec{r}|$$

where $f(r)$ is some function.

Then,

$$\vec{r} \times \vec{F} = f\vec{r} \times \hat{r} = 0 \quad \Rightarrow \quad \dot{\vec{L}} = 0$$



i.e. the angular momentum is conserved.

• **Gravitational force** is (approximately) central:

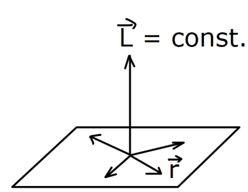
$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$

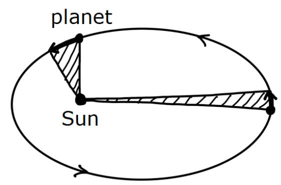
$$\vec{L} = \vec{r} \times \vec{p} = \text{const.}$$

0.4.2 Consequences of conserved angular momentum

• $\vec{L}$ is constant and always perpendicular to $\vec{r}$.

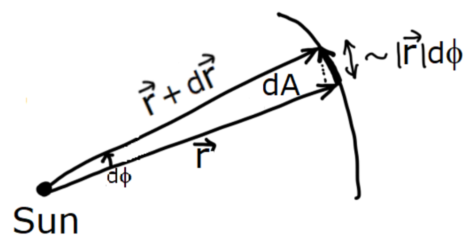
Hence, motion lies in a plane.





• **Kepler's 2nd law:**

The line between the Sun and the planet sweeps equal areas in equal times.



Infinitesimal area:

$$dA = \frac{1}{2}(\text{base}) \times (\text{height}) = \frac{1}{2}|\vec{r}| \times |\vec{r}|d\phi = \frac{1}{2}r^2 d\phi$$

Let us divide by dt. We want to show that $\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\phi}{dt}$ is constant.

On the other hand,

$$\vec{L} = \vec{r} \times m\dot{\vec{r}}$$

We can write

$$\dot{\vec{r}} \approx \frac{(\vec{r}+d\vec{r}) - \vec{r}}{dt}$$

Using this we have

$$\vec{L}dt = \vec{r} \times m\left((\vec{r}+d\vec{r}) - \vec{r}\right)$$

We drop the second term (since $\vec{r} \times \vec{r} = 0$)

$$\vec{L}dt = \vec{r} \times m(\vec{r}+d\vec{r})$$

Take the absolute value

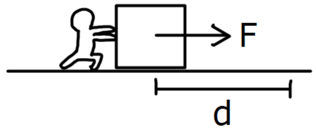
$$|\vec{L}dt| = m|\vec{r}|\underbrace{|\vec{r}+d\vec{r}|}_{\approx|\vec{r}|}\underbrace{\sin d\phi}_{\approx d\phi} \approx mr^2 d\phi$$

Thus,

$$|\vec{L}| = mr^2\frac{d\phi}{dt}$$

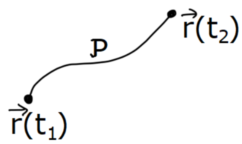
Since $\vec{L}$ is constant, $\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\phi}{dt}$ is also constant. *QED*

0.5 Work



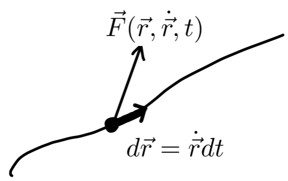
"Force applied throughout a distance is work," i.e. $W = F \times d$

What is the work done by the force $\vec{F}$ when a particle moves along a path $\mathcal{P}$?



$$\boxed{W[\mathcal{P}] = \int_{\mathcal{P}} \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} \vec{F} \cdot \dot{\vec{r}}\, dt}$$

The force can depend on $\vec{r}$, $\dot{\vec{r}}$, t:



Use Newton's 2nd law $\vec{F} = \dot{\vec{p}} = m\dot{\vec{v}}$

$$W[\mathcal{P}] = \int_{t_1}^{t_2} m\dot{\vec{v}} \cdot \vec{v}\, dt = \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{1}{2}m\vec{v}^2\right) dt = \frac{1}{2}m\vec{v}^2\bigg|_{t=t_2} - \frac{1}{2}m\vec{v}^2\bigg|_{t=t_1}$$

Define the **kinetic energy**

$$\boxed{T \equiv \frac{1}{2}m\vec{v}^2 = \frac{\vec{p}^2}{2m}}$$

Then,

$$\boxed{W = T_2 - T_1}$$

i.e. the kinetic energy changes by the work done.

0.6 Conservative force

The force $\vec{F}$ is said to be **conservative** if it can be expressed as

$$\boxed{\vec{F} = -\vec{\nabla} V}$$

where $V(\vec{r})$ is a scalar function called a **potential**.

• Note: because $\vec{\nabla}(\text{const}) = 0$, we can always add an arbitrary constant to V without changing $\vec{F}$ (i.e. can freely choose the zero level of V).

• Newton's 2nd law becomes:

$$\boxed{\dot{\vec{p}} = -\vec{\nabla} V}$$

• If $\vec{F}$ is conservative, then the work done by $\vec{F}$ between points $\vec{r}_1$ and $\vec{r}_2$ is

$$W_{12} = \int_{\vec{r}_1}^{\vec{r}_2} d\vec{r} \cdot \vec{F} = -\int_{\vec{r}_1}^{\vec{r}_2} d\vec{r} \cdot \vec{\nabla} V = -V(\vec{r}_2) + V(\vec{r}_1)$$

by the fundamental theorem of calculus (which is $\int_1^2 dx \frac{d}{dx} f = f_2 - f_1$).

Thus, W_{12} is **independent of the path**!

• If $\vec{r}_1 = \vec{r}_2$ (i.e. closed path), then $W = 0$: no net work has been done.

• The converse is also true:

If work is independent of the path, then we can *define* a potential

$$V(\vec{r}) \equiv -\int_{\vec{r}_0}^{\vec{r}} d\vec{r}' \cdot \vec{F}(\vec{r}')$$

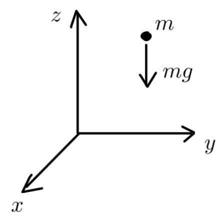
where $\vec{r}_0$ is some fixed reference point. Then this quantity will obey

$$\nabla V = -\vec{F}$$

again by the fundamental theorem of calculus. Hence, the force is conservative.

0.7 Examples

0.7.1 Particle in a gravitational field



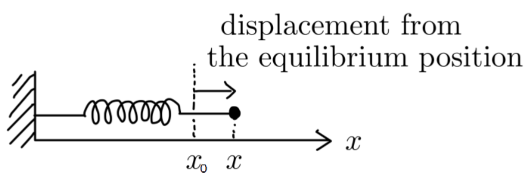
$$\vec{F} = (0,\, 0,\, -mg)$$

This can be derived from the potential:

$$V(\vec{r}) = +mgz$$

Indeed: $-\vec{\nabla} V = (0,\, 0,\, -mg) = \vec{F}$.

0.7.2 Elastic force



Hooke's law:

$$F = -k(x - x_0)$$

This can be derived from the potential

$$V(x) = \frac{1}{2} k (x - x_0)^2$$

0.8 Energy conservation with conservative forces

For a conservative force

$$W = V_1 - V_2$$

We also saw that

$$W = T_2 - T_1$$

Thus,

$$\boxed{T_1 + V_1 = T_2 + V_2}$$

which means that the **total energy**

$$\boxed{E \equiv T + V = \frac{\vec{p}^2}{2m} + V(\vec{r})}$$

is conserved!

• One can prove conservation (t-independence) of E more directly:

$$\dot{E} = \frac{d}{dt}\left(\frac{\vec{p}^2}{2m} + V(\vec{r})\right) = \frac{1}{m}\vec{p}\cdot\dot{\vec{p}} + \vec{\nabla}V \cdot \dot{\vec{r}} = \vec{v}\cdot\vec{F} + \vec{\nabla}V \cdot \vec{v} = \vec{v}\cdot \underbrace{\left(\vec{F} + \vec{\nabla}V\right)}_{\substack{\equiv 0 \text{ for a} \\ \text{conservative} \\ \text{system}}} = 0$$

• Mechanical systems whose energy is conserved are called **conservative systems**.